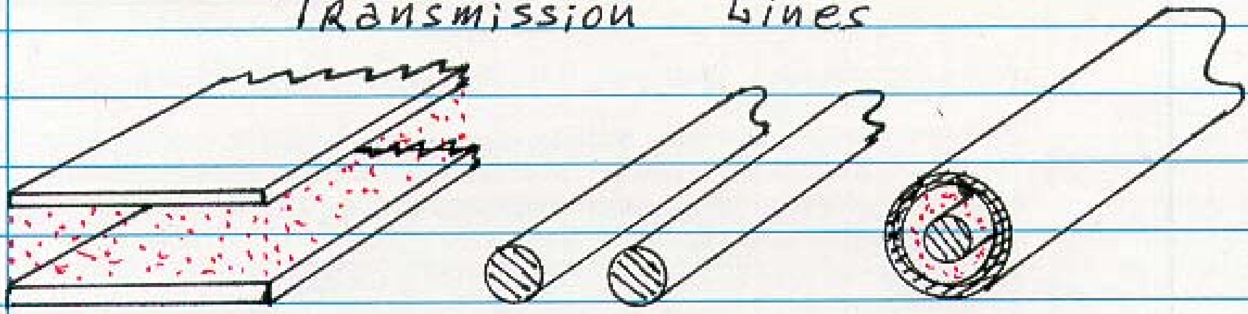
Review of Transmission Lines

References

- David K. Cheng, Fundamentals of Engineering Electromagnetics, Addison Wesley, Chapter 8, 1993.

- William H. Hayt and John A. Buck, Engineering Electromagnetics, Sixth Edition, McGraw Hill, Chapter 13, 2001.

Transmission Lines



Parallel-plate "Striplines" — Two-wire "Flat or twisted pair" — Coaxial "Cable"

TEM waves may be guided (in one direction) in the dielectric medium between two conductors.

Assume: ⓐ The distance between the conductors is small compared to the wavelength λ.

ⓑ The length of the conductors is long compared to the wavelength λ.

ⓒ An external source causes a cosinuoidal electric/magnetic field in time (i.e. $\cos \omega t$ or $Re(e^{j\omega t})$).

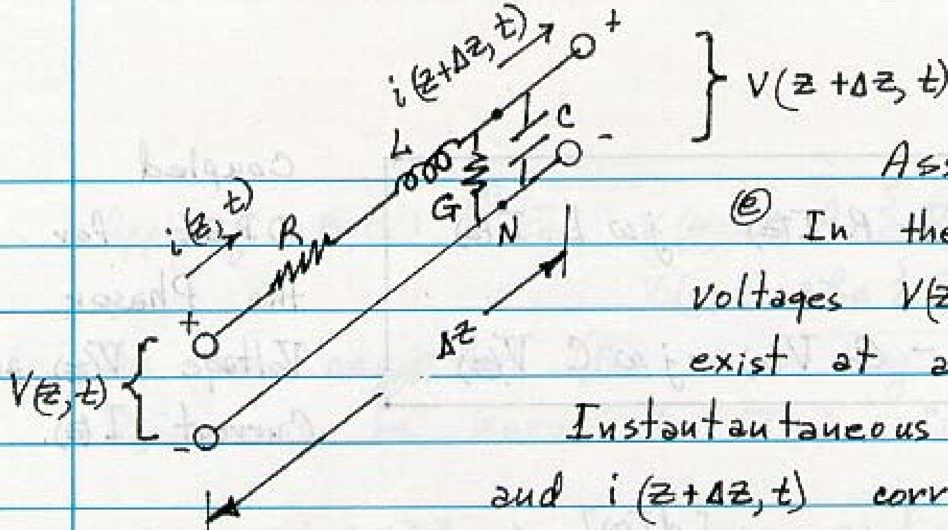
ⓓ Four parameters are characteristic of the line:

R = resistance per unit length (in both conductors) $\frac{\Omega}{m}$

L = inductance " " " (" " ") $\frac{H}{m}$

G = conductance " " " $\frac{S}{m}$

C = capacitance " " " $\frac{F}{m}$



Assume:

ⓔ In the transmission line voltages $V(z,t)$ and $V(z+\Delta z,t)$ exist at an instant in time. Instantantaneous currents $i(z,t)$ and $i(z+\Delta z,t)$ correspond.

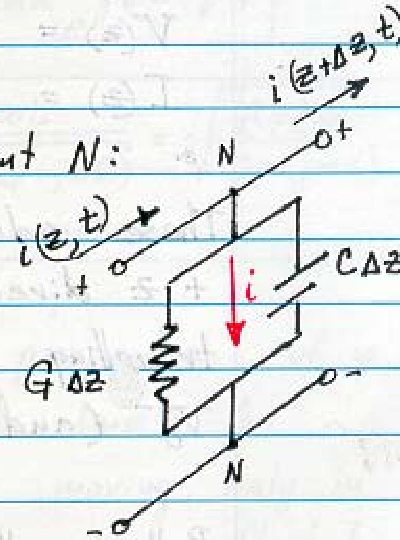
Then applying Kirchoff's voltage law:

$$V(z,t) - R\,\Delta z\; i(z,t) - L\,\Delta z\,\frac{\partial i(z,t)}{\partial t} = V(z+\Delta z,t) \quad \text{or}$$

$$\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -R\, i(z,t) - L\,\frac{\partial i(z,t)}{\partial t} \quad \text{or}$$

$$\boxed{\frac{\partial V(z,t)}{\partial z} = -R\, i(z,t) - L\,\frac{\partial i(z,t)}{\partial t}}$$

Applying Kirchoff's current law at point N:



$$i(z,t) - G\,\Delta z\; V(z+\Delta z,t) - C\,\Delta z\,\frac{\partial V(z+\Delta z,t)}{\partial t} = i(z+\Delta z,t) \quad \text{or}$$

$$\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G\, V(z,t) - C\,\frac{\partial V(z,t)}{\partial t} \quad \text{or}$$

$$\boxed{\frac{\partial i(z,t)}{\partial z} = -G\, V(z,t) - C\,\frac{\partial V(z,t)}{\partial t}}$$

← General Transmission Line Equations

$$V(z,t) = \mathrm{Re}\left\{ V(z)\, e^{j\omega t} \right\}$$
$$i(z,t) = \mathrm{Re}\left\{ I(z)\, e^{j\omega t} \right\}$$

so

$$\frac{dV(z)}{dz} = -R\,I(z) - j\omega L\, I(z)$$

$$\frac{dI(z)}{dz} = -G\,V(z) - j\omega C\, V(z)$$

Coupled ODE's for the Phasor Voltage $V(z)$ and Current $I(z)$.

$$\therefore \frac{d^2V(z)}{dz^2} = -(R+j\omega L)\left[\frac{dI(z)}{dz}\right] = (R+j\omega L)(G+j\omega C)\,V(z) \text{ and}$$

$$\frac{d^2I(z)}{dz^2} = (G+j\omega C)(R+j\omega L)\,I(z) \quad \text{OR}$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \; ; \quad \frac{d^2I(z)}{dz^2} = \gamma^2 I(z) \; ; \quad \gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

propagation constant

2nd Order, Linear, ODEs with solutions

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \; ; \quad v(z,t) = Re\left\{V(z)e^{j\omega t}\right\}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \; ; \quad i(z,t) = Re\left\{I(z)e^{j\omega t}\right\}$$

These solutions represent travelling waves in the $+z$ direction of amplitude V_0^+ (or I_0^+) and travelling waves in the $-z$ direction of amplitude V_0^- (and I_0^-).

Putting these solutions back into the Phasor equations

$$V_0^+(-\gamma)e^{-\gamma z} + V_0^- \gamma e^{\gamma z} = -(R+j\omega L)\left[I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}\right] \text{ or}$$

$$\frac{V_0^+}{I_0^+} = \frac{R+j\omega L}{\gamma} = Z_0 \text{ and } \frac{V_0^-}{I_0^-} = -\frac{R+j\omega L}{\gamma} = -Z_0$$

Applying B.C.: (a) For $V(0,t) = Re\{\tilde{V}(0)\, e^{j\omega t}\}$ and $V(\infty,t) = Re\{\tilde{V}(\infty)\, e^{j\omega t}\}$

we can see that (for an ∞-long transmission line) $\tilde{V}_o^-$ must be zero if $\tilde{V}(\infty)$ = finite. Thus

$$\tilde{V}(z) = \tilde{V}_o^+ e^{-\gamma z} \quad \text{and} \quad V(z,t) = Re\{\tilde{V}_o^+ e^{j\omega t} e^{-\gamma z}\}$$
$$= Re\{\tilde{V}_o^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\}$$

α = attenuation constant $\frac{Np}{m}$ ← neper = dimensionless (page 289)

β = phase constant $\frac{rad}{m}$

Similarly for current B.C. on an ∞ long transmission line

$$\tilde{I}(z) = \tilde{I}_o^+ e^{-\gamma z} \quad \text{so} \quad i(z,t) = Re\{\tilde{I}_o^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\}$$

Defining
$$Z_o \equiv \frac{\tilde{V}_o^+}{\tilde{I}_o^+} = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(R+j\omega L)(G+j\omega C)}} = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$$
= characteristic impedance Ω

Note: γ and Z_o are characteristic properties of a finite or infinite transmission line. An infinitely long line has waves moving only in the z-direction (i.e. there are no reflected waves).

1. Special Case: Lossless Line ($R=0$, $G=0$)

a. propagation constant $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$ ∴

$\gamma = \sqrt{-\omega^2 LC} = j\omega\sqrt{LC}$; $\alpha = 0$, $\beta = \omega\sqrt{LC}$

↑ a linear function of ω

b. phase velocity u_p = velocity of $e^{j(\omega t - \beta z)} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$

compare this to the phase velocity of a TEM wave in a medium $u_p = \frac{1}{\sqrt{\mu\epsilon}}$ and we see the equivalent $\boxed{LC = \mu\epsilon}$

using this equivalence, if we can find L, C is determined (and vice versa).

c. Characteristic impedance $Z_o = R_o + jX_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}$

$R_o = \sqrt{\frac{L}{C}}$ (constant) and $X_o = 0$

2. Special Case: Distortionless Line ($\frac{R}{L} = \frac{G}{C}$)

a. propagation constant $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(\frac{RC}{L} + j\omega C)}$ or

$\gamma = \sqrt{\frac{C}{L}}(R + j\omega L)$; $\alpha = R\sqrt{\frac{C}{L}}$ and $\beta = \omega\sqrt{LC}$ as above

b. phase velocity $u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$ (constant)

c. Characteristic impedance $Z_o = R_o + jX_o = \sqrt{\frac{R+j\omega L}{\frac{RC}{L} + j\omega C}} = \sqrt{\frac{L}{C}}$

$R_o = \sqrt{\frac{L}{C}}$ (constant) and $X_o = 0$ (as above).

Note: Except for the attenuation constant α, the characteristics of a distortionless line are the same as a loss-less line.

Note: Because $u_p = \frac{1}{\sqrt{LC}}$ is a constant for all frequencies, ω, then a signal pulse [which is described by a Fourier transform (page 54) in frequency space] will not broaden or distort its shape as it travels down the transmission line. Hence the name "distortionless line."

Note: For a lossy transmission line the propagation constant $\gamma = \alpha + i\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$ will (in general) not depend linearly upon ω. Thus it will lead to a frequency dependent u_p. As different frequency components of a signal pulse propagate along the transmission Line (with different velocities) the signal pulse will broaden (i.e. suffer distortion or dispersion). A lossy transmission line (with non-zero R and/or G) is therefore dispersive.

(see Cheng page 304)

$Re\{V(z,t)\}Re\{i(z,t)\} = P(z,t)$ = power propagated along z. ∴

$$P(z,t) = Re\left[V_o^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\right] Re\left[I_o^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\right]$$

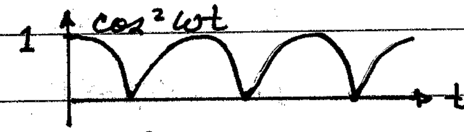
$\uparrow \frac{V_o^+}{Z_o} \leftarrow$ complex $\therefore \frac{V_o^+}{R_o + jX_o} = \frac{V_o^+}{|Z_o|^2} R$

$$P(z,t) = \left\{\frac{V_o^{+2}}{|Z_o|^2} R_o e^{-2\alpha z} Re[\cos\omega t + j\sin\omega t][\cos\beta z - j\sin\beta z] \times Re[\cos\omega t + j\sin\omega t][\cos\beta z - j\sin\beta z]\right\}$$

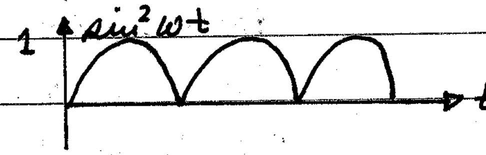
$$= \frac{V_o^2}{|Z_o|^2} R_o e^{-2\alpha z}[\cos\omega t\cos\beta z + \sin\omega t\sin\beta z]^2$$

$$\therefore \quad P(z,t) = \frac{V_o^2}{|Z_o|^2} R_o \, e^{-2\alpha z} \left[\cos^2\omega t \cos^2\beta z + \sin^2\omega t \sin^2\beta z + 2 \sin\omega t \cos\omega t \sin\beta z \cos\beta z \right]$$

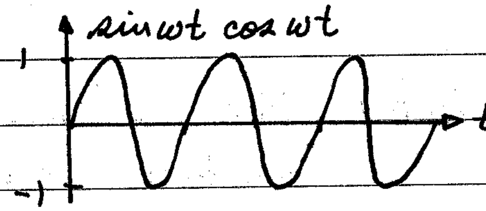
The first term goes as



The second term goes as



The third term goes as



$$\therefore \; P(z)_{Av} = \int_0^T P(z,t)\, dt = \left[\frac{1}{2} \cos^2\beta z + \frac{1}{2} \sin^2\beta z + 0 \right] \frac{V_o^2}{|Z_o|^2} R_o e^{-2\alpha z}$$

or

$$\boxed{P_{Av}(z) = \frac{1}{2} \frac{V_o^2}{|Z_o|^2} R_o \, e^{-2\alpha z}}$$

Note: The power being propagated along z is decreasing, i.e. power is being lost with z. The lost power $P_L(z) = -\frac{\partial P(z)}{\partial z} = 2\alpha P_{Av}(z)$ or

$$\boxed{\alpha = \frac{P_L(z)}{2 P_{Av}(z)}}$$

Note: Power is lost in resistive heating, either along z by $[I(t)]^2 R$ or across the medium by $[V(t)]^2 G$. i.e.

$$P_L(z,t) = \left[Re\left\{ I(z)\, e^{+j\omega t} \right\} \right]^2 R + \left[Re\left\{ V(z)\, e^{+j\omega t} \right\} \right]^2 G$$

as on page 105 $Re\{V_o^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\}$

$= V_o^+ e^{-\alpha z} \; Re\{[\cos\omega t + j\sin\omega t][\cos\beta z - j\sin\beta z]\}$

$= V_o^+ e^{-\alpha z} [\cos\omega t \cos\beta z + \sin\omega t \sin\beta z]$

so that when we square this term and average over one cycle (integrate over T) we get

← $\frac{1}{2} V_o^{+2} e^{-2\alpha z}$

likewise $Re\{I_o^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\}$ with $I_o^+ = \frac{V_o^+}{Z_o}$ ← comp

becomes $\frac{V_o^+}{R_o + jX_o}$

and averaging over one cycle we get for the square

$$\frac{1}{2} \frac{V_o^{+2}}{|Z_o|^2} e^{-2\alpha z}$$

$$\therefore P_{L_{AV}}(z) = \frac{V_o^{+2}}{2|Z_o|^2} R e^{-2\alpha z} + \frac{V_o^{+2}}{2} G e^{-2\alpha z} \quad \text{or}$$

$$= \frac{V_o^{+2}}{2|Z_o|^2}\left[R + G|Z_o|^2\right] e^{-2\alpha z} \quad \text{so}$$

$$\alpha = \frac{P_L(z)}{2P_{av}(z)} = \frac{1}{2} \frac{[R + G|Z_o|^2]}{R_o}$$

Special Case: If the line is loss-less $Z_o = R_o + jX_o = \sqrt{\frac{L}{C}} + j\,0$

so if the line is low-loss $Z_o \approx \sqrt{\frac{L}{C}}$ and

$$\alpha \approx \frac{1}{2}\left(\frac{R}{R_o} + \frac{G}{R_o}\frac{L}{C}\right) = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

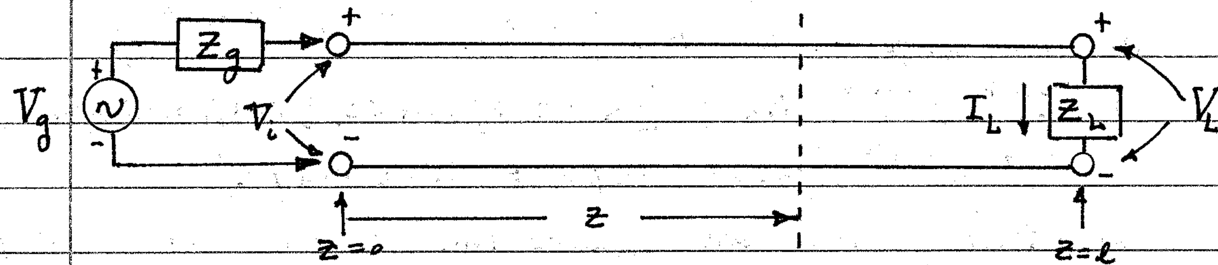
(page 104)

Special Case: If the line is distortionless $Z_o = R_o + jX_o = \sqrt{\frac{L}{C}}$

and $\left(\frac{R}{L} = \frac{G}{C}\right)$ so $\alpha = \frac{1}{2}\left(\frac{R}{R_o} + \frac{G}{R_o}\frac{L}{C}\right) = \frac{1}{2}\left(\frac{R}{R_o} + \frac{R}{R_o}\right) = \frac{R}{R_o}$ or

$$\alpha = R\sqrt{\frac{C}{L}}$$

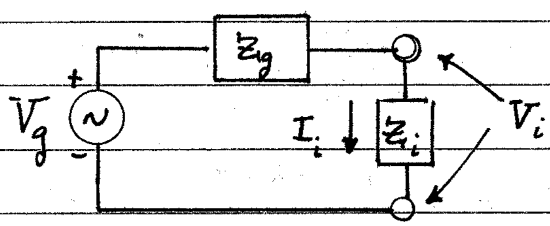
Finite Transmission Lines



Suppose a voltage generator produces a sinusoidal voltage $V_g \cos \omega t$ and it has internal impedance Z_g.

Suppose a load impedance (e.g. your TV set) with impedance Z_L is located at $z = \ell$. Thus at $z = \ell$

$$\frac{V_L}{I_L} = Z_L$$



Then the generator sees an "input impedance" Z_i with $\frac{V_i}{I_i} = Z_i$ = "line impedance" + "load impedance"

We found (page 102) that in the transmission line

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \quad \text{where } V(z,t) = Re\{V(z) e^{j\omega t}\}$$

$$I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z} \quad \text{and } \frac{V_o^+}{I_o^+} = \frac{R + j\omega L}{\gamma}; \quad \frac{V_o^-}{I_o^-} = \frac{-R - j\omega L}{\gamma}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \uparrow Z_o \quad\quad\quad\quad \uparrow -Z_o$

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Applying B.C. at $z = \ell$

$$V(\ell) = V_o^+ e^{-\gamma \ell} + V_o^- e^{\gamma \ell} = V_L$$

$$I(\ell) = I_o^+ e^{-\gamma \ell} + I_o^- e^{\gamma \ell} = I_L \quad \text{or}$$

$$I(\ell) = \frac{V_o^+}{Z_o} e^{-\gamma \ell} - \frac{V_o^-}{Z_o} e^{\gamma \ell} = I_L \quad \text{and solving for } V_o^+ \text{ and } V_o^-$$

$\uparrow$ line impedance

$$V_o^+ = \frac{1}{2}\left(V_L + I_L Z_o\right) e^{\gamma l} = \frac{I_L}{2}\left(Z_L + Z_o\right) e^{\gamma l}$$
$$V_o^- = \frac{1}{2}\left(V_L - I_L Z_o\right) e^{-\gamma l} = \frac{I_L}{2}\left(Z_L - Z_o\right) e^{-\gamma l}$$

and putting these coefficients back into the transmission equations

$$V(z) = \frac{I_L}{2}\left[(Z_L + Z_o) e^{\gamma(l-z)} + (Z_L - Z_o) e^{-\gamma(l-z)}\right]$$
$$I(z) = \frac{I_L}{2 Z_o}\left[(Z_L + Z_o) e^{\gamma(l-z)} - (Z_L - Z_o) e^{-\gamma(l-z)}\right]$$

and if we let $(l-z) = z'$

$$V(z') = \frac{I_L}{2}\left[(Z_L + Z_o) e^{\gamma z'} + (Z_L - Z_o) e^{-\gamma z'}\right]$$
$$I(z') = \frac{I_L}{2 Z_o}\left[(Z_L + Z_o) e^{\gamma z'} - (Z_L - Z_o) e^{-\gamma z'}\right]$$

but

$$e^{\gamma z'} + e^{-\gamma z'} = 2 \cosh \gamma z' \quad \text{and} \quad e^{\gamma z'} - e^{-\gamma z'} = 2 \sinh \gamma z'$$

$$V(z') = I_L \left(Z_L \cosh \gamma z' + Z_o \sinh \gamma z'\right)$$
$$I(z') = \frac{I_L}{Z_o}\left(Z_L \sinh \gamma z' + Z_o \cosh \gamma z'\right)$$

Now lets define $Z(z') = \frac{V(z')}{I(z')}$ as the line impedance at z'

$$\therefore \; Z(z') = Z_o \frac{Z_L \cosh \gamma z' + Z_o \sinh \gamma z'}{Z_L \sinh \gamma z' + Z_o \cosh \gamma z'} = Z_o \frac{Z_L + Z_o \tanh \gamma z'}{Z_o + Z_L \tanh \gamma z'}$$

and evaluating this at the generator end $z = 0$ (or $z' = l$)

$$\boxed{Z_i = Z_o \frac{Z_L + Z_o \tanh \gamma l}{Z_o + Z_L \tanh \gamma l} \quad \Omega}$$

This is the impedance the generator "sees".

Note: If $Z_L = Z_o$, $Z_i = Z_o$ regardless of the length l.
i.e. if we choose $Z_L = Z_o$ the source sees an impedance of Z_o (it is as if the transmission line were not present). Under the condition $Z_L = Z_o$ we say the line is "matched".

Note: For loss-less transmission lines $R=0$, $G=0$

$$\gamma = \alpha + i\beta = \sqrt{(R+j\omega L)(G+j\omega C)} = j\omega\sqrt{LC}$$

i.e. $\alpha = 0$ and $\beta = \omega\sqrt{LC}$ in which case

$$Z_i = Z_o \frac{Z_L + Z_o \tanh(j\omega\sqrt{LC}\,l)}{Z_o + Z_L \tanh(j\omega\sqrt{LC}\,l)}$$ and since

$\tanh jx = j \tan x$

$$Z_i = Z_o \frac{Z_L + Z_o j \tan(\omega\sqrt{LC}\,l)}{Z_o + Z_L j \tan(\omega\sqrt{LC}\,l)}$$ with $Z_o = R_o$ (page 104)

$$\boxed{Z_i = R_o \frac{Z_L + j R_o \tan(\omega\sqrt{LC}\,l)}{R_o + j Z_L \tan(\omega\sqrt{LC}\,l)}, \quad R_o = \sqrt{\frac{L}{C}}}$$

Special Case: If $Z_L \Rightarrow \infty$ Open-Circuit Line

$$Z_i = -jR_o \frac{1}{\tan(\omega\sqrt{LC}\,l)} = -jR_o \cot(\omega\sqrt{LC}\,l)$$

Here the line impedance is purely reactive and

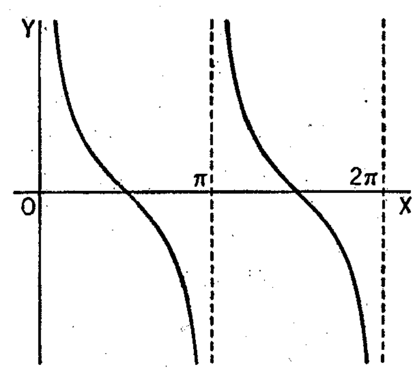
Z_i can be $\pm \rightarrow$

Let's let $u_p = \frac{1}{\sqrt{LC}}$

$\therefore \omega \frac{1}{u_p} l = 2\pi f\left(\frac{1}{u_p}\right) l$

$= \left(\frac{2\pi}{T u_p}\right) l = \frac{2\pi}{\lambda} l$ so

$$Z_i = -j\sqrt{\frac{L}{C}} \cot\left(2\pi \frac{l}{\lambda}\right)$$



COTANGENT CURVE

$y = \text{ctn}\, x$

Very Special Case: If $l \ll \lambda$

$$\cot 2\pi\frac{l}{\lambda} = \frac{1}{\tan\left(2\pi\frac{l}{\lambda}\right)} \cong \frac{1}{2\pi\frac{l}{\lambda}}$$

so

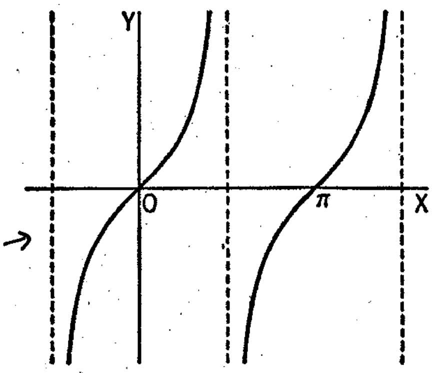
$$Z_i \cong -j\sqrt{\frac{L}{C}} \frac{\lambda}{2\pi l} = -j\frac{\sqrt{L/C}}{\omega\sqrt{LC}\,l} = -j\frac{1}{\omega C l}$$

which is the impedance of a capacitor of Cl farads (remember C = capacitance per unit length here). ✓

Special Case: If $Z_L = 0$ Short Circuit

$Z_i = j R_o \tan(\omega\sqrt{CL}\, \ell)$

which is also purely reactive and can be ± →



TANGENT CURVE
$y = \tan x$

Very Special Case: If $\omega\sqrt{LC}\,\ell = \frac{\pi}{2}$ →

$Z_i = j R_o \infty$ or $-j R_o \infty$

∴ if $2\pi \frac{\ell}{\lambda} = \frac{\pi}{2}$ i.e. $\ell = \frac{\lambda}{4}$

$Z_i \to \infty$ (effectively open circuit)

For quarter wavelength transmission lines (or multiples ↗) a short circuit acts as if there were no cable present.

Very Special Case: If $\ell << \lambda$ $\tan(\omega\sqrt{CL}\,\ell) \approx 2\pi\frac{\ell}{\lambda}$

$Z_i \cong j R_o \, \omega\sqrt{LC}\,\ell = j\sqrt{\frac{L}{C}}\,\omega\sqrt{LC}\,\ell = j\omega(L\ell)$

which is the impedance of a pure inductor (remember L = inductance per unit length here) ✓

Note: We may use Open and Closed circuit measurements to find values of an unknown transmission line Z_o & γ:

Open Circuit ($Z_L \to \infty$) (p. 109) gives $Z_{io} = \frac{Z_o}{\tanh \gamma\ell}$

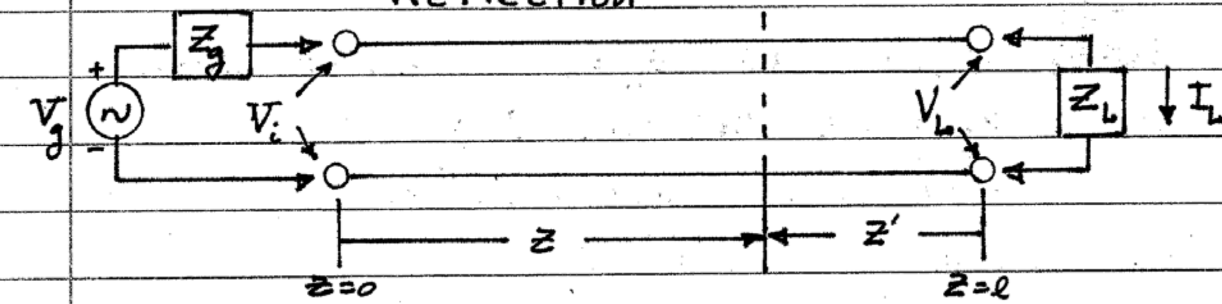
Closed Circuit ($Z_L \to 0$) " " $Z_{is} = Z_o \tanh \gamma\ell$

∴ $Z_{io} Z_{is} = Z_o^2$ and $\gamma = \frac{1}{\ell}\tanh^{-1}\frac{Z_{is}}{Z_o} = \frac{1}{\ell}\tanh^{-1}\frac{Z_{is}}{\sqrt{Z_{is} Z_{io}}}$

$$\boxed{Z_o = \sqrt{Z_{io} Z_{is}} \quad \text{and} \quad \gamma = \frac{1}{\ell}\tanh^{-1}\sqrt{\frac{Z_{is}}{Z_{io}}}}$$

Measure Z_{is} & Z_{io} to find Z_o & γ.

Reflection



$$V(z') = \frac{I_L}{2}\left[(Z_L + Z_0)\, e^{\gamma z'} + (Z_L - Z_0)\, e^{-\gamma z'}\right] \quad (p. 109)$$

$$I(z') = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)\, e^{\gamma z'} - (Z_L - Z_0)\, e^{-\gamma z'}\right]$$

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}, \quad Z_0 = R_0 + jX_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

(p. 102) (p. 103)

$$V(z',t) = Re\left\{V(z')\, e^{j\omega t}\right\} = Re\left\{\frac{I_L}{2}\left[(Z_L+Z_0)\, e^{\alpha z'} e^{j(\omega t + \beta z')} + \ldots\right.\right.$$

$$\text{\& } I(z',t) = Re\left\{I(z')\, e^{j\omega t}\right\}$$

incident wave ($z' = \ell - z$)

+ reflected wave

$$\therefore \boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma|\, e^{j\theta_r}}$$

= voltage reflection coefficient of the load impedance Z_L.

and since we called $V_0^+ = \frac{I_L}{2}(Z_L + Z_0)$, $V_0^- = \frac{I_L}{2}(Z_L - Z_0)$

(p. 102)

$$\Gamma = \frac{V_0^-}{V_0^+}$$ is a complex number with $|\Gamma| \leq 1$.

Note: For a "matched" load ($Z_L = Z_0$) $|\Gamma| = 0$

so there is no reflected wave

Note: For $Z_L \neq Z_0$, $|\Gamma| \neq 0$ and the voltage equations produce standing waves with maxima & minima.

$$\boxed{S = \frac{|V_{max}|}{|V_{min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

(see page 74) & $|\Gamma| = \frac{S-1}{S+1}$

13.5 SEVERAL PRACTICAL PROBLEMS

In this section we shall direct our attention to two examples of practical transmission line problems. The first is the determination of load impedance from experimental data, and the second is the design of a single-stub matching network.

Let us assume that we have made experimental measurements on a 50-Ω air line which show that there is a standing wave ratio of 2.5. This has been determined by moving a sliding carriage back and forth along the line to determine maximum and minimum voltage readings. A scale provided on the track along which the carriage moves indicates that a *minimum* occurs at a scale reading of 47.0 cm, as shown in Fig. 13.12. The zero point of the scale is arbitrary and does not correspond to the location of the load. The location of the minimum is usually specified instead of the maximum because it can be determined more accurately than that of the maximum; think of the sharper minima on a rectified sine wave. The frequency of operation is 400 MHz, so the wavelength is 75 cm. In order to pinpoint the location of the load, we remove it and replace it with a short circuit; the position of the minimum is then determined as 26.0 cm.

We know that the short circuit must be located an integral number of half-wavelengths from the minimum; let us arbitrarily locate it one half-wavelength away at $26.0 - 37.5 = -11.5$ cm on the scale. Since the short circuit has replaced the load, the load is also located at -11.5 cm. Our data thus show that the minimum is $47.0 - (-11.5) = 58.5$ cm from the load, or subtracting one-half wavelength, a minimum is 21.0 cm from the load. The voltage *maximum* is thus $21.0 - (37.5/2) = 2.25$ cm from the load, or $2.25/75 = 0.030$ wavelength from the load.

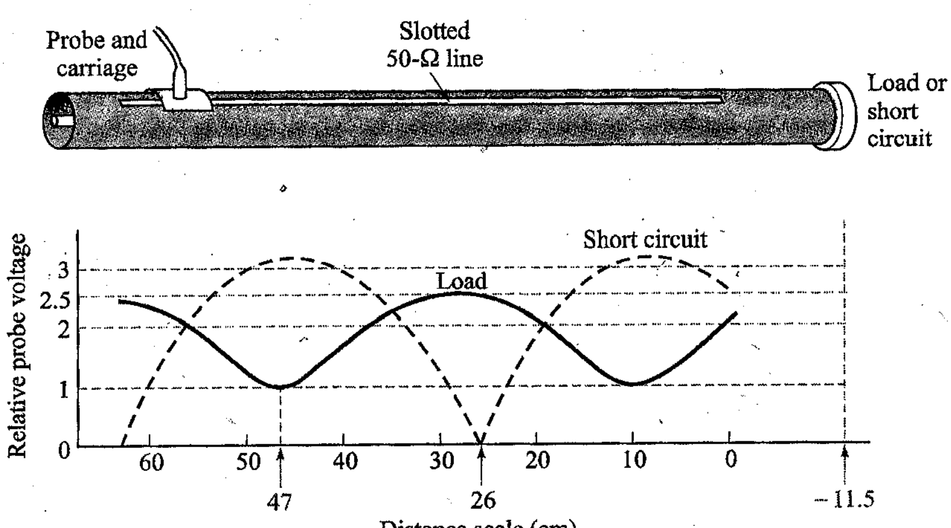
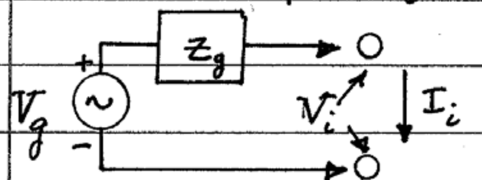


FIGURE 13.12
A sketch of a coaxial slotted line. The distance scale is on the slotted line. With the load in place, $s = 2.5$, and the minimum occurs at a scale reading of 47 cm; for a short circuit the minimum is located at a scale reading of 26 cm. The wavelength is 75 cm.

Generator Perspective



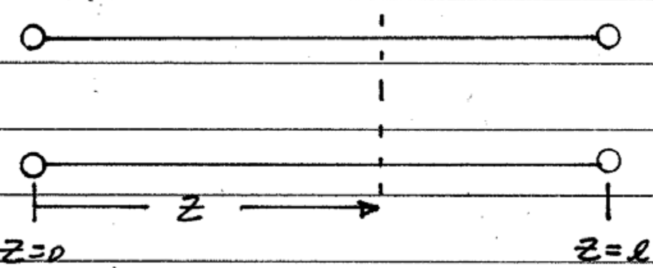
From the generator's perspective

$$V_i = V_g - I_i Z_g$$

so the input voltage, V_i, to the transmission line is

$$V_i = V_g \frac{Z_o}{(Z_o + Z_g)} \quad \text{(potentiometer ratio)}$$

This voltage travels down the transmission Line



$V^+(z,t) = R_e\{ V_o^+ \; e^{-\alpha z} e^{j(\omega t - \beta z)}\}$ (p. 108)

until at the load ($z = \ell$)

$V^+(\ell, t) = R_e \{ V_o^+ e^{-\alpha \ell} e^{j(\omega t - \beta \ell)}\}$

here, the reflected voltage is

$V^-(\ell, t) = Re\{ V_o^- e^{\alpha \ell} e^{j(\omega t + \beta \ell)}\}$ with $\frac{V_o^-}{V_o^+} = \Gamma$ and

$V^-(z', \ell) = Re\{ V_o^- e^{-\alpha z'} e^{j(\omega t - \beta z')}\}$ will move in the z' direction

Thus (unless $|\Gamma| = 0$ i.e. a "matched" load) the reflected wave travels (at velocity $u_p = \frac{\omega}{\beta}$) back toward $z = 0$ ($z' = \ell$). When it arrives back at the generator it will have amplitude $Re\{\Gamma V_o^+ e^{-2\alpha \ell}\}$. This wave will then be reflected by the generator impedance with a reflection coefficient $\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o}$.

Special Cases: $Z_L = Z_o$ (matched load) there is only an incident wave.
$Z_g = Z_o$ ($Z_L \neq Z_o$) there is only one reflected wave.

13.3 SOME TRANSMISSION-LINE EXAMPLES

In this section we shall apply many of the results that we have obtained in the previous two sections to several typical transmission-line problems. We shall simplify our work by restricting our attention to the lossless line.

Let us begin by assuming a two-wire 300-Ω line ($Z_0 = 300\ \Omega$), such as the lead-in wire from the antenna to a television or FM receiver. The circuit is shown in Fig. 13.5. The line is 2 m long and the dielectric constant is such that the velocity on the line is 2.5×10^8 m/s. We shall terminate the line with a receiver having an input resistance of 300 Ω and represent the antenna by its Thevenin equivalent $Z_{Th} = 300\ \Omega$ in series with $V_{s,Th} = 60$ V at 100 MHz. This antenna voltage is larger by a factor of about 10^5 than it would be in a practical case, but it also provides simpler values to work with; in order to think practical thoughts, divide currents or voltages by 10^5, divide powers by 10^{10}, and leave impedances alone.

Since the load impedance is equal to the characteristic impedance, the line is matched; the reflection coefficient is zero, and the standing wave ratio is unity. For the given velocity and frequency, the wavelength on the line is $v/f = 2.5$ m, and the phase constant is $2\pi/\lambda = 0.8\pi$ rad/m; the attenuation constant is zero. The electrical length of the line is $\beta l = (0.8\pi)2$, or 1.6π rad. This length may also be expressed as 288°, or 0.8 wavelength.

The input impedance offered to the voltage source is 300 Ω, and since the internal impedance of the source is 300 Ω, the voltage at the input to the line is half of 60 V, or 30 V. The source is matched to the line and delivers the maximum available power to the line. Since there is no reflection and no attenuation, the voltage at the load is 30 V, but it is delayed in phase by 1.6π rad. Thus

$$V_{in} = 30 \cos(2\pi 10^8 t) \quad \text{V}$$

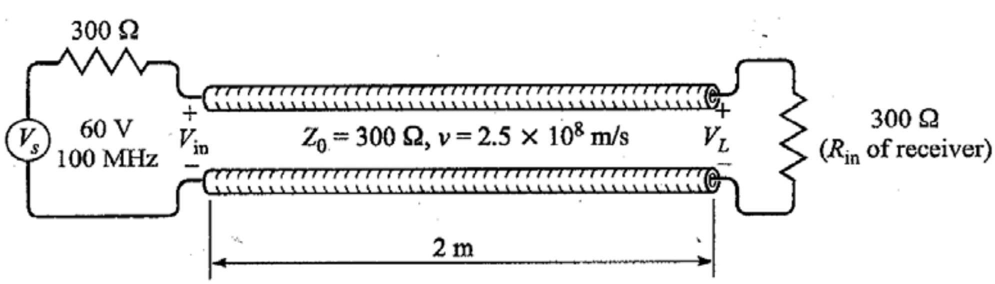


FIGURE 13.5
A transmission line that is matched at each end produces no reflections and thus delivers maximum power to the load.

whereas

$$V_L = 30\cos(2\pi 10^8 t - 1.6\pi) \quad \text{V}$$

The input current is

$$I_{in} = \frac{V_{in}}{300} = 0.1\cos(2\pi 10^8 t) \quad \text{A}$$

while the load current is

$$I_L = 0.1\cos(2\pi 10^8 t - 1.6\pi) \quad \text{A}$$

The average power delivered to the input of the line by the source must all be delivered to the load by the line,

$$P_{in} = P_L = \frac{1}{2} \times 30 \times 0.1 = 1.5 \quad \text{W}$$

Now let us connect a second receiver, also having an input resistance of 300 Ω, across the line in parallel with the first receiver. The load impedance is now 150 Ω, the reflection coefficient is

$$\Gamma = \frac{150 - 300}{150 + 300} = -\frac{1}{3}$$

and the standing wave ratio on the line is

$$s = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = 2$$

The input impedance is no longer 300 Ω, but is now

$$Z_{in} = Z_0 \frac{Z_L \cos \beta l + jZ_0 \sin \beta l}{Z_0 \cos \beta l + jZ_L \sin \beta l} = 300 \frac{150 \cos 288^\circ + j300 \sin 288^\circ}{300 \cos 288^\circ + j150 \sin 288^\circ}$$
$$= 510\angle{-23.8^\circ} = 466 - j206 \quad \Omega$$

which is a capacitive impedance. Physically, this means that this length of line stores more energy in its electric field than in its magnetic field. The input current phasor is thus

$$I_{s,in} = \frac{60}{300 + 466 - j206} = 0.0756\angle 15.0^\circ \quad \text{A}$$

and the power supplied to the line by the source is

$$P_{in} = \frac{1}{2} \times (0.0756)^2 \times 466 = 1.333 \quad \text{W}$$

Since there are no losses in the line, 1.333 W must also be delivered to the load. Note that this is less than the 1.50 W which we were able to deliver to a matched load; moreover, this power must divide equally between two receivers, and thus each receiver now receives only 0.667 W. Since the input impedance of each receiver is 300 Ω, the voltage across the receiver is easily found as

$$0.667 = \frac{1}{2}\frac{|V_{s,L}|^2}{300}$$

$$|V_{s,L}| = 20 \quad \text{V}$$

in comparison with the 30 V obtained across the single load.

Before we leave this example, let us ask ourselves several questions about the voltages on the transmission line. Where is the voltage a maximum and a minimum, and what are these values? Does the phase of the load voltage still differ from the input voltage by 288°? Presumably, if we can answer these questions for the voltage, we could do the same for the current.

We answered questions of this nature for the uniform plane wave in the last chapter, and our analogy should therefore provide us with the corresponding information for the transmission line. In Sec. 12.2, Eq. (21) serves to locate the voltage maxima at

$$z_{max} = -\frac{1}{2\beta}(\phi + 2m\pi) \quad (m = 0, 1, 2, \ldots)$$

where $\Gamma = |\Gamma|e^{j\phi}$. Thus, with $\beta = 0.8\pi$ and $\phi = \pi$, we find

$$z_{max} = -0.625 \text{ and } -1.875 \quad \text{m}$$

while the minima are $\lambda/4$ distant from the maxima,

$$z_{min} = 0 \text{ and } -1.25 \quad \text{m}$$

and we find that the load voltage (at $z = 0$) is a voltage minimum. This, of course, verifies the general conclusion we reached in the last chapter: a voltage minimum occurs at the load if $Z_L < Z_0$, and a voltage maximum occurs if $Z_L > Z_0$, where both impedances are pure resistances.

The minimum voltage on the line is thus the load voltage, 20 V; the maximum voltage must be 40 V, since the standing wave ratio is 2. The voltage at the input end of the line is

$$V_{s,in} = I_{s,in} Z_{in} = (0.0756\angle 15.0°)(510\angle -23.8°) = 38.5\angle -8.8°$$

The input voltage is almost as large as the maximum voltage anywhere on the line because the line is about three-quarters wavelength long, a length which would place the voltage maximum at the input when $Z_L < Z_0$.

The final question we posed for ourselves deals with the relative phase of the input and load voltages. Although we have found each of these voltages, we do not know the phase angle of the load voltage. From Sec. 12.2, Eq. (18), the voltage at any point on the line is

$$V_s = \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)V_0^+ \tag{35}$$

We may use this expression to determine the voltage at any point on the line in terms of the voltage at any other point. Since we know the voltage at the input to the line, we let $z = -l$,

$$V_{s,in} = \left(e^{j\beta l} + \Gamma e^{-j\beta l}\right) V_0^+ \tag{36}$$

and solve for V_0^+,

$$V_0^+ = \frac{V_{s,in}}{e^{j\beta l} + \Gamma e^{-j\beta l}} = \frac{38.5\angle{-8.8^\circ}}{e^{j1.6\pi} - \frac{1}{3}e^{-j1.6\pi}} = 30.0\angle 72.0^\circ \quad \text{V}$$

We may now let $z = 0$ in (35) to find the load voltage,

$$V_{s,L} = (1 + \Gamma)V_0^+ = 20\angle 72^\circ = 20\angle{-288^\circ}$$

The amplitude agrees with our previous value. The presence of the reflected wave causes $V_{s,in}$ and $V_{s,L}$ to differ in phase by about -279° instead of -288°.

Example 13.2

In order to provide a slightly more complicated example, let us now place a purely capacitive impedance of $-j300\ \Omega$ in parallel with the two 300-Ω receivers. We are to find the input impedance and the power delivered to each receiver.

Solution. The load impedance is now 150 Ω in parallel with $-j300\ \Omega$, or

$$Z_L = \frac{150(-j300)}{150 - j300} = \frac{-j300}{1 - j2} = 120 - j60 \quad \Omega$$

We first calculate the reflection coefficient and the standing wave ratio:

$$\Gamma = \frac{120 - j60 - 300}{120 - j60 + 300} = \frac{-180 - j60}{420 - j60} = 0.447\angle{-153.4^\circ}$$

$$s = \frac{1 + 0.447}{1 - 0.447} = 2.62$$

Thus, the standing wave ratio is higher and the mismatch is therefore worse. Let us next calculate the input impedance. The electrical length of the line is still 288°, so that

$$Z_{in} = 300\frac{(120 - j60)\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(120 - j60)\sin 288^\circ} = 755 - j138.5 \quad \Omega$$

This leads to a source current of

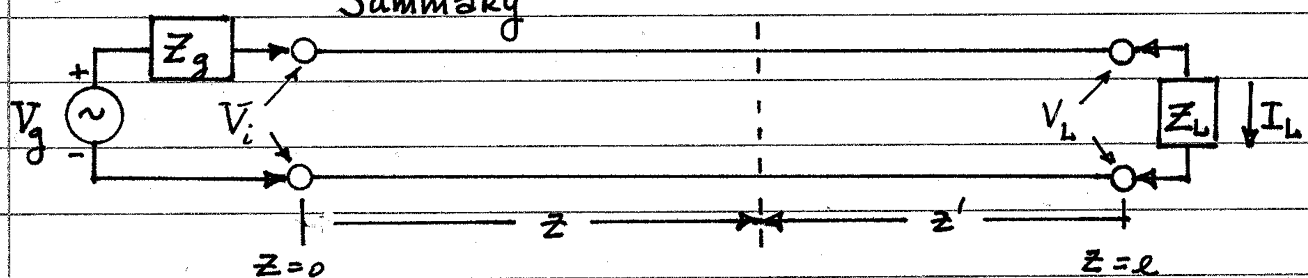
$$I_{s,in} = \frac{V_{Th}}{Z_{Th} + Z_{in}} = \frac{60}{300 + 755 - j138.5} = 0.0564\angle 7.47^\circ \quad \text{A}$$

Therefore, the average power delivered to the input of the line is $P_{in} = \frac{1}{2}(0.0564)^2(755) = 1.200$ W. Since the line is lossless, it follows that $P_L = 1.200$ W, and each receiver gets only 0.6 W.

Example 13.3

As a final example let us terminate our line with a purely capacitive impedance, $Z_L = -j300\ \Omega$. We seek the reflection coefficient, the standing-wave ratio, and the power delivered to the load.

Summary



Generator	Transmission Line	Load

Transmission Line

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \qquad \frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

$$\gamma \equiv \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)} = \text{propagation constant}$$

$$\left.\begin{aligned} V(z) &= V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \\ I(z) &= I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z} \end{aligned}\right\} \quad Z_o = \frac{V_o^+}{I_o^+}, \quad Z_o = -\frac{V_o^-}{I_o^-}$$

$$Z_o = \frac{R+j\omega L}{\gamma} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = R_o + jX_o$$

Load

$$Z_L = R_L + jX_L$$

$$\frac{Z_L}{Z_o} = z_L$$

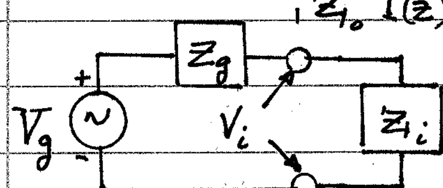
$$z_L = r + jx$$

$$z_L = \frac{R_L + jX_L}{Z_o}$$

Transmission Line + Load

$$V(z') = \frac{I_L}{2}(Z_L + Z_o)e^{\gamma z'} + \frac{I_L}{2}(Z_L - Z_o)e^{-\gamma z'}$$

$$Z_o I(z') = \frac{I_L}{2}(Z_L + Z_o)e^{\gamma z'} - \frac{I_L}{2}(Z_L - Z_o)e^{-\gamma z'}$$



$$Z_i = Z(z=0) = Z(z'=\ell) = \frac{V(z'=\ell)}{I(z'=\ell)}$$

$$Z_i = Z_o \frac{(Z_L+Z_o)e^{\gamma \ell} + (Z_L - Z_o)e^{-\gamma \ell}}{(Z_L+Z_o)e^{\gamma \ell} - (Z_L - Z_o)e^{-\gamma \ell}} = Z_o \frac{1 + \Gamma e^{-2\gamma \ell}}{1 - \Gamma e^{-2\gamma \ell}}$$

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \Gamma_r + j\Gamma_i = |\Gamma| e^{j\theta_r} = \frac{\frac{Z_L}{Z_o} - 1}{\frac{Z_L}{Z_o} + 1} = \frac{z_L - 1}{z_L + 1} \quad \text{\& } z_L = \frac{1+\Gamma}{1-\Gamma}$$

$$z_L = r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} \cdot \frac{1-\Gamma_r + j\Gamma_i}{1-\Gamma_r + j\Gamma_i} = \frac{(1+\Gamma_r)(1-\Gamma_r) - \Gamma_i^2 + j\Gamma_i(1-\Gamma_r) + j\Gamma_i(1+\Gamma_r)}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$\therefore \quad r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad \left\{\begin{matrix} 8\text{-}102 \text{ \& } 8\text{-}103 \\ 13\text{-}41 \text{ \& } 13\text{-}42 \end{matrix}\right\}$$

13.4 GRAPHICAL METHODS

Transmission line problems often involve manipulations with complex numbers, making the time and effort required for a solution several times greater than that needed for a similar sequence of operations on real numbers. One means of reducing the labor without seriously affecting the accuracy is by using transmission-line charts. Probably the most widely used one is the Smith chart.[4]

Basically, this diagram shows curves of constant resistance and constant reactance; these may represent either an input impedance or a load impedance. The latter, of course, is the input impedance of a zero-length line. An indication of location along the line is also provided, usually in terms of the fraction of a wavelength from a voltage maximum or minimum. Although they are not specifically shown on the chart, the standing-wave ratio and the magnitude and angle of the reflection coefficient are very quickly determined. As a matter of fact, the diagram is constructed within a circle of unit radius, using polar coordinates, with radius variable $|\Gamma|$ and counterclockwise angle variable ϕ, where $\Gamma = |\Gamma|e^{j\phi}$. Figure 13.6 shows this circle. Since $|\Gamma| < 1$, all our information must lie on or within the unit circle. Peculiarly enough, the reflection coefficient itself will not be plotted on the final chart, for these additional contours would make the chart very difficult to read. $\phi = \theta_p$

After several lines of elementary algebra, we may write (41) and (42) in forms which readily display the nature of the curves on Γ_r, Γ_i axes,

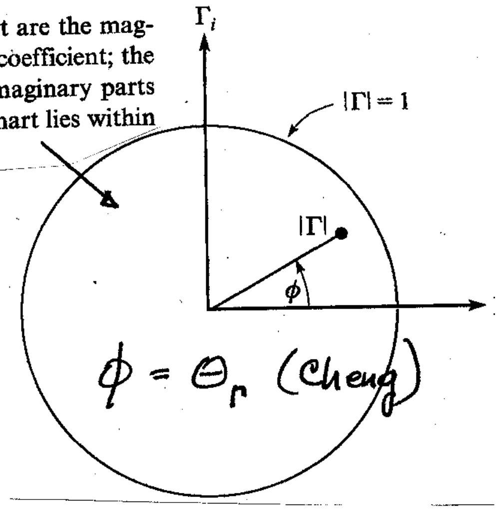
$$\boxed{\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2} \tag{43}$$

$$\boxed{(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2} \tag{44}$$

The first equation describes a family of circles, where each circle is associated with a specific value of resistance r. For example, if $r = 0$ the radius of this zero-resistance circle is seen to be unity, and it is centered at $\Gamma_r = 0$, $\Gamma_i = 0$, the origin. This checks, for a pure reactance termination leads to a reflection coefficient of unity magnitude. On the other hand, if $r = \infty$, then $z_L = \infty$ and we have $\Gamma = 1 + j0$. The circle described by (43) is centered at $\Gamma_r = 1$, $\Gamma_i = 0$ and has zero radius. It is therefore the point $\Gamma = 1 + j0$, as we decided it should be. As another example, the circle for $r = 1$ is centered at $\Gamma_r = 0.5$, $\Gamma_i = 0$ and has a radius of 0.5. This circle is shown on Fig. 13.7, along with circles for $r = 0.5$ and $r = 2$. All circles are centered on the Γ_r axis and pass through the point $\Gamma = 1 + j0$.

Equation (44) also represents a family of circles, but each of these circles is defined by a particular value of x, rather than r. If $x = \infty$, then $z_L = \infty$, and $\Gamma = 1 + j0$ again. The circle described by (44) is centered at $\Gamma = 1 + j0$ and has zero radius; it is therefore the point $\Gamma = 1 + j0$. If $x = +1$, then the circle is

FIGURE 13.6
The polar coordinates of the Smith chart are the magnitude and phase angle of the reflection coefficient; the cartesian coordinates are the real and imaginary parts of the reflection coefficient. The entire chart lies within the unit circle $|\Gamma| = 1$.



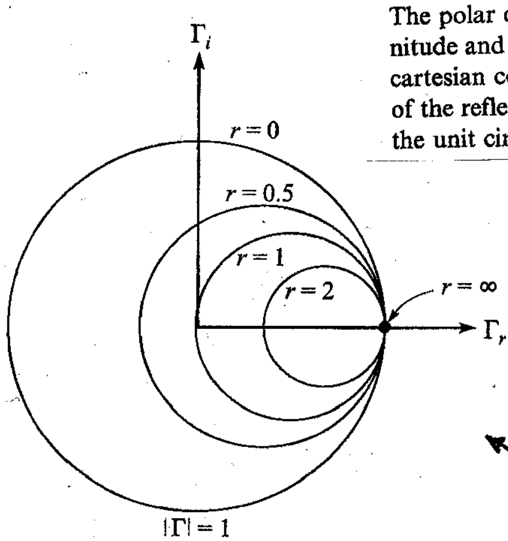


FIGURE 13.7
Constant-r circles are shown on the Γ_r, Γ_i plane. The radius of any circle is $1/(1+r)$.

centered at $\Gamma = 1 + j1$ and has unit radius. Only one-quarter of this circle lies within the boundary curve $|\Gamma| = 1$, as shown in Fig. 13.8. A similar quarter-circle appears below the Γ_r axis for $x = -1$. The portions of other circles for $x = 0.5$, -0.5, 2, and -2 are also shown. The "circle" representing $x = 0$ is the Γ_r axis; this is also labeled on Fig. 13.8.

The two families of circles both appear on the Smith chart, as shown in Fig. 13.9. It is now evident that if we are given Z_L, we may divide by Z_0 to obtain z_L, locate the appropriate r and x circles (interpolating as necessary), and determine Γ by the intersection of the two circles. Since the chart does not have concentric circles showing the values of $|\Gamma|$, it is necessary to measure the radial distance from the origin to the intersection with dividers or compass and use an auxiliary scale to find $|\Gamma|$. The graduated line segment below the chart in Fig. 13.9 serves this purpose. The angle of Γ is ϕ, and it is the counter-clockwise angle from the Γ_r axis. Again, radial lines showing the angle would clutter up the chart badly, so

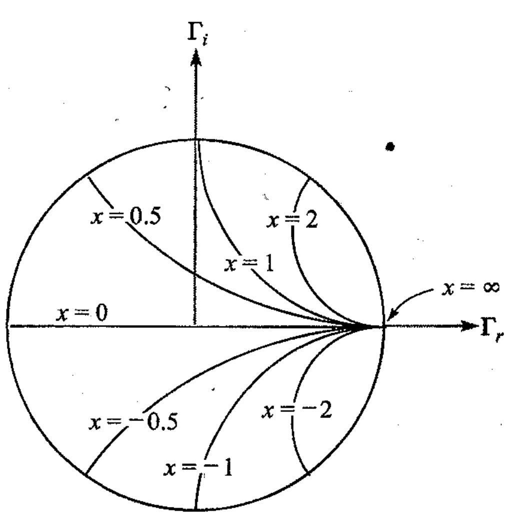


FIGURE 13.8
The portions of the circles of constant x lying within $|\Gamma| = 1$ are shown on the Γ_r, Γ_i axes. The radius of a given circle is $1/|x|$.

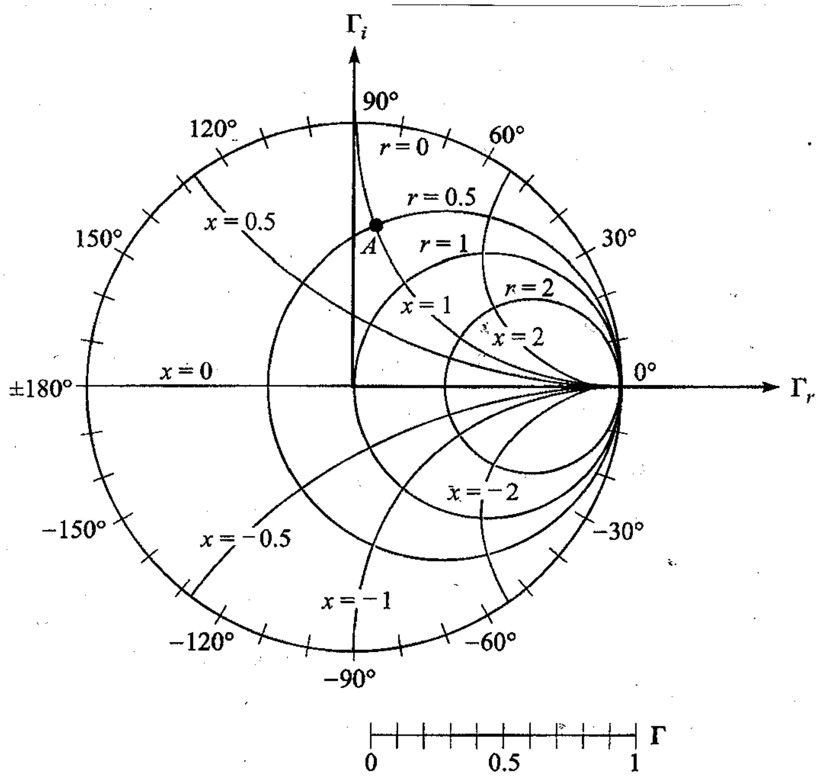


FIGURE 13.9
The Smith chart contains the constant-r circles and constant-x circles, an auxiliary radial scale to determine $|\Gamma|$, and an angular scale on the circumference for measuring ϕ.

the angle is indicated on the circumference of the circle. A straight line from the origin through the intersection may be extended to the perimeter of the chart. As an example, if $Z_L = 25 + j50\ \Omega$ on a 50-Ω line, $z_L = 0.5 + j1$, and point A on Fig. 13.9 shows the intersection of the $r = 0.5$ and $x = 1$ circles. The reflection coefficient is approximately 0.62 at an angle ϕ of 83°.

The Smith chart is completed by adding a second scale on the circumference by which distance along the line may be computed. This scale is in wavelength units, but the values placed on it are not obvious. To obtain them, we first divide the voltage at any point along the line,

$$V_s = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$$

by the current

$$I_s = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$$

obtaining the normalized input impedance

$$z_{in} = \frac{V_s}{Z_0 I_s} = \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}$$

Replacing z by $-l$ and dividing numerator and denominator by $e^{j\beta l}$, we have the general equation relating normalized input impedance, reflection coefficient, and line length,

$$z_{in} = \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} = \frac{1 + |\Gamma| e^{j(\phi - 2\beta l)}}{1 - |\Gamma| e^{j(\phi - 2\beta l)}} \tag{45}$$

Note that when $l = 0$, we are located at the load, and $z_{in} = (1 + \Gamma)/(l - \Gamma) = z_L$, as shown by (38).

Equation (45) shows that the input impedance at any point $z = -l$ can be obtained by replacing Γ, the reflection coefficient of the load, by $\Gamma e^{-j2\beta l}$. That is, we decrease the angle of Γ by $2\beta l$ radians as we move from the load to the line input. Only the angle of Γ is changed; the magnitude remains constant.

Thus, as we proceed from the load z_L to the input impedance z_{in}, we move *toward* the generator a distance l on the transmission line, but we move through a *clockwise* angle of $2\beta l$ on the Smith chart. Since the magnitude of Γ stays constant, the movement toward the source is made along a constant-radius circle. One lap around the chart is accomplished whenever βl changes by π rad, or when l changes by one-half wavelength. This agrees with our earlier discovery that the input impedance of a half-wavelength lossless line is equal to the load impedance.

The Smith chart is thus completed by the addition of a scale showing a change of 0.5λ for one circumnavigation of the unit circle. For convenience, two scales are usually given, one showing an increase in distance for clockwise movement and the other an increase for counterclockwise travel. These two scales are shown in Fig. 13.10. Note that the one marked "wavelengths toward generator" (wtg) shows increasing values of l/λ for clockwise travel, as described above. The zero point of the wtg scale is rather arbitrarily located to the left. This corresponds to input impedances having phase angles of 0° and $R_L < Z_0$. We have also seen that voltage minima are always located here.

Example 13.4

The use of the transmission line chart is best shown by example. Let us again consider a load impedance, $Z_L = 25 + j50\ \Omega$, terminating a 50-Ω line. The line length is 60 cm and the operating frequency is such that the wavelength on the line is 2 m. We desire the input impedance.

Solution. We have $z_L = 0.5 + j1$, which is marked as A on Fig. 13.11, and we read $\Gamma = 0.62\angle 82°$. By drawing a straight line from the origin through A to the circumference, we note a reading of 0.135 on the wtg scale. We have $l/\lambda = 0.6/2 = 0.3$, and it is therefore 0.3λ from the load to the input. We therefore find z_{in} on the $|\Gamma| = 0.62$ circle opposite a wtg reading of $0.135 + 0.300 = 0.435$. This construction is shown in Fig. 13.11, and the point locating the input impedance is marked B. The normalized input impedance is read as $0.28 - j0.40$, and thus $Z_{in} = 14 - j20$. A more accurate analytical calculation gives $Z_{in} = 13.7 - j20.2$.

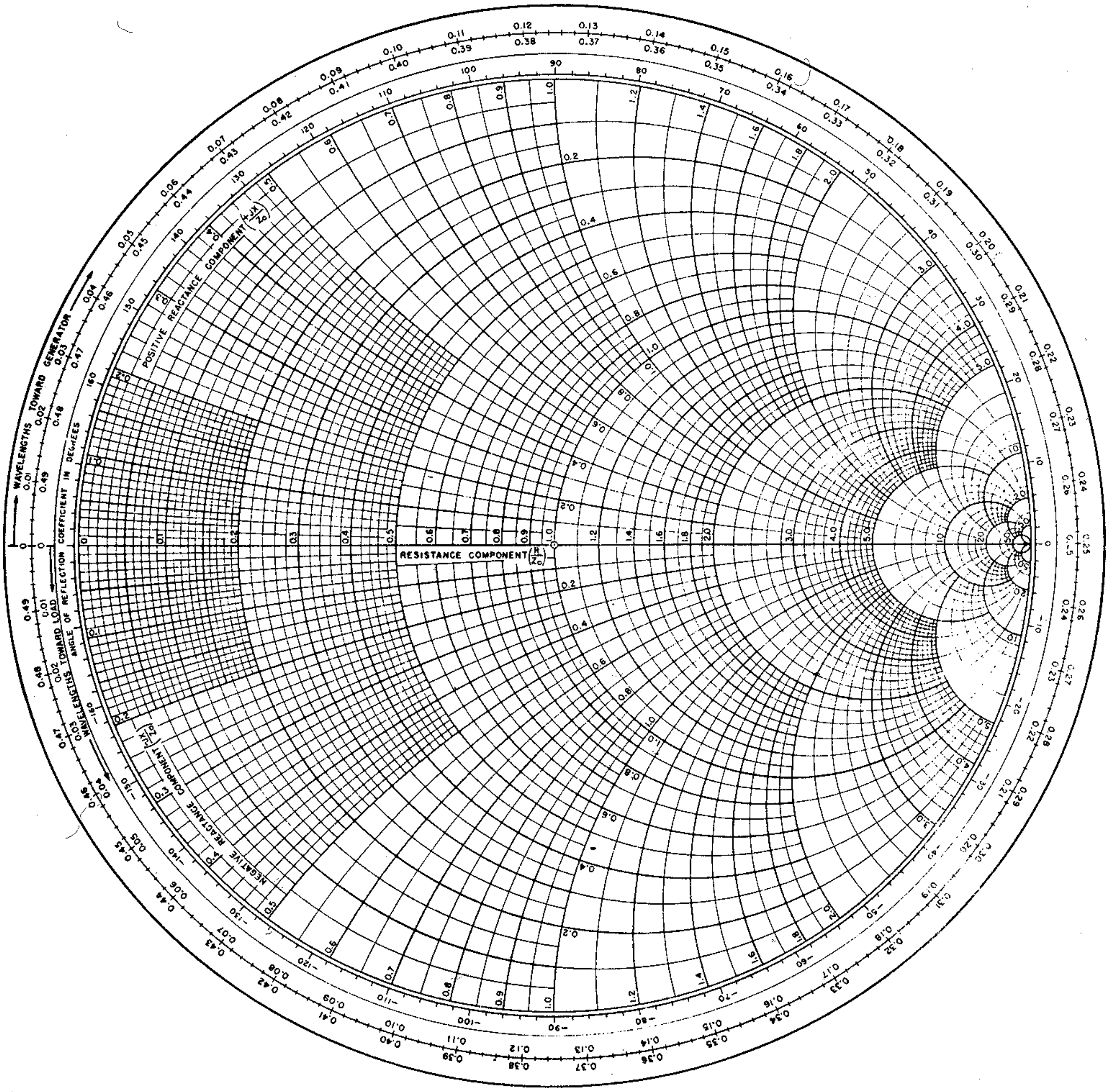


FIGURE 13.10
A photographic reduction of one version of a useful Smith chart (*courtesy of the Emeloid Company, Hillside, N.J.*). For accurate work, larger charts are available wherever fine technical books are sold.

Information concerning the location of the voltage maxima and minima is also readily obtained on the Smith chart. We already know that a maximum or minimum must occur at the load when Z_L is a pure resistance; if $R_L > Z_0$ there is a maximum at the load, and if $R_L < Z_0$ there is a minimum. We may extend this result now by noting that we could cut off the load end of a transmission line at a point where the input impedance is a pure resistance and replace that section with a resistance R_{in}; there would be no changes on the generator portion of the line. It follows, then, that the location of voltage maxima and minima must be at those points where Z_{in} is a pure resistance. Purely resistive input impedances

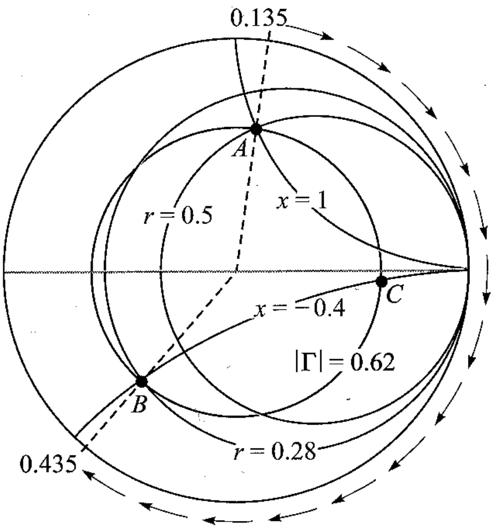


FIGURE 13.11
The normalized input impedance produced by a normalized load impedance $z_L = 0.5 + j1$ on a line 0.3λ long is $z_{in} = 0.28 - j0.40$.

must occur on the $x = 0$ line (the Γ_r axis) of the Smith chart. Voltage maxima or current minima occur when $r > 1$, or at wtg $= 0.25$, and voltage minima or current maxima occur when $r < 1$, or at wtg $= 0$. In the example above, then, the maximum at wtg $= 0.250$ must occur $0.250 - 0.135 = 0.115$ wavelengths toward the generator from the load. This is a distance of 0.115×200, or 23 cm from the load.

We should also note that since the standing wave ratio produced by a resistive load R_L is either R_L/R_0 or R_0/R_L, whichever is greater than unity, the value of s may be read directly as the value of r at the intersection of the $|\Gamma|$ circle and the r axis, $r > 1$. In our example this intersection is marked point C, and $r = 4.2$; thus, $s = 4.2$.

Transmission line charts may also be used for normalized admittances, although there are several slight differences in such use. We let $y_L = Y_L/Y_0 = g + jb$ and use the r circles as g circles and the x circles as b circles. The two differences are: first, the line segment where $g > 1$ and $b = 0$ corresponds to a voltage minimum; and second, 180° must be added to the angle of Γ as read from the perimeter of the chart. We shall use the Smith chart in this way in the following section.

Special charts are also available for non-normalized lines, particularly 50-Ω charts and 20-mS charts.

✔ **D13.7.** A load $Z_L = 80 - j100\ \Omega$ is located at $z = 0$ on a lossless 50-Ω line. The operating frequency is 200 MHz and the wavelength on the line is 2 m. (a) If the line is 0.8 m in length, use the Smith chart to find the input impedance. (b) What is s? (c) What is the distance from the load to the nearest voltage maximum? (d) What is the distance from the input to the nearest point at which the remainder of the line could be replaced by a pure resistance?

Ans. $79 + j99\ \Omega$: 4.50; 0.0397 m; 0.760 m

13.5 SEVERAL PRACTICAL PROBLEMS

In this section we shall direct our attention to two examples of practical transmission line problems. The first is the determination of load impedance from experimental data, and the second is the design of a single-stub matching network.

Let us assume that we have made experimental measurements on a 50-Ω air line which show that there is a standing wave ratio of 2.5. This has been determined by moving a sliding carriage back and forth along the line to determine maximum and minimum voltage readings. A scale provided on the track along which the carriage moves indicates that a *minimum* occurs at a scale reading of 47.0 cm, as shown in Fig. 13.12. The zero point of the scale is arbitrary and does not correspond to the location of the load. The location of the minimum is usually specified instead of the maximum because it can be determined more accurately than that of the maximum; think of the sharper minima on a rectified sine wave. The frequency of operation is 400 MHz, so the wavelength is 75 cm. In order to pinpoint the location of the load, we remove it and replace it with a short circuit; the position of the minimum is then determined as 26.0 cm.

We know that the short circuit must be located an integral number of half-wavelengths from the minimum; let us arbitrarily locate it one half-wavelength away at $26.0 - 37.5 = -11.5$ cm on the scale. Since the short circuit has replaced the load, the load is also located at -11.5 cm. Our data thus show that the minimum is $47.0 - (-11.5) = 58.5$ cm from the load, or subtracting one-half wavelength, a minimum is 21.0 cm from the load. The voltage *maximum* is thus $21.0 - (37.5/2) = 2.25$ cm from the load, or $2.25/75 = 0.030$ wavelength from the load.

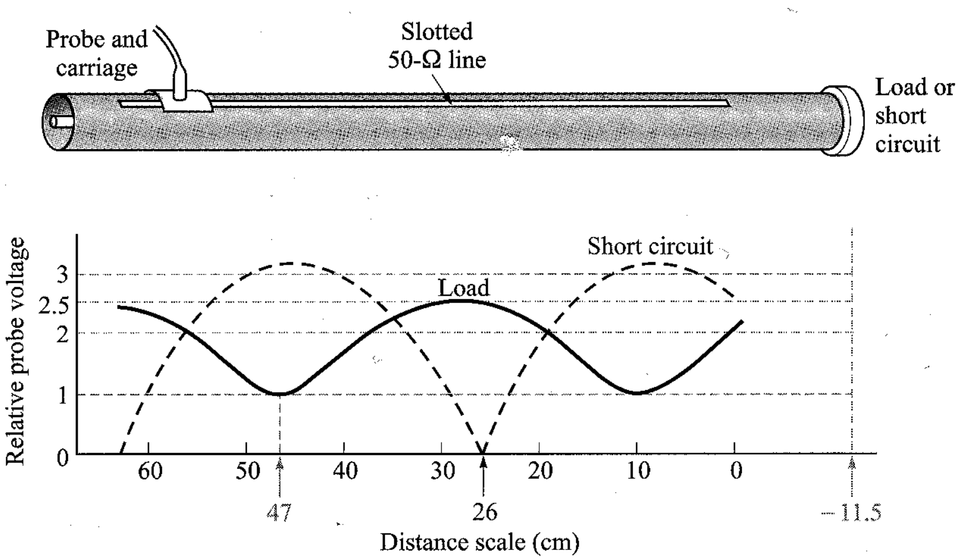


FIGURE 13.12
A sketch of a coaxial slotted line. The distance scale is on the slotted line. With the load in place, $s = 2.5$, and the minimum occurs at a scale reading of 47 cm; for a short circuit the minimum is located at a scale reading of 26 cm. The wavelength is 75 cm.

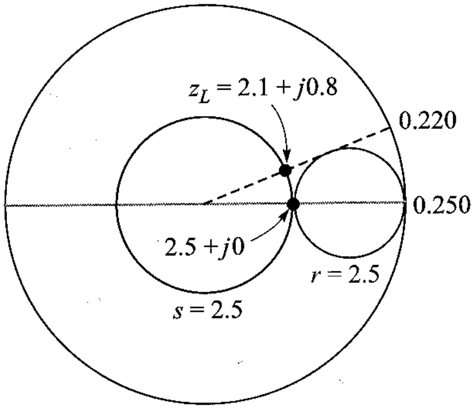


FIGURE 13.13
If $z_{in} = 2.5 + j0$ on a line 0.03 wavelength long, then $z_L = 2.1 + j0.8$.

With this information, we can now turn to the Smith chart. At a voltage maximum the input impedance is a pure resistance equal to sR_0; on a normalized basis, $z_{in} = 2.5$. We therefore enter the chart at $z_{in} = 2.5$ and read 0.250 on the wtg scale. Subtracting 0.030 wavelength to reach the load, we find that the intersection of the $s = 2.5$ (or $|\Gamma| = 0.429$) circle and the radial line to 0.220 wavelength is at $z_L = 2.1 + j0.8$. The construction is sketched on the Smith chart of Fig. 13.13. Thus $Z_L = 105 + j40\,\Omega$, a value which assumes its location at a scale reading of -11.5 cm, or an integral number of half-wavelengths from that position. Of course, we may select the "location" of our load at will by placing the short circuit at that point which we wish to consider as the load location. Since load locations are not well defined, it is important to specify the point (or plane) at which the load impedance is determined.

As a final example, let us try to match this load to the 50-Ω line by placing a short-circuited stub of length d_1 a distance d from the load (see Fig. 13.14). The stub line has the same characteristic impedance as the main line. The lengths d and d_1 are to be determined.

The input impedance to the stub is a pure reactance; when combined in parallel with the input impedance of the length d containing the load, the resul-

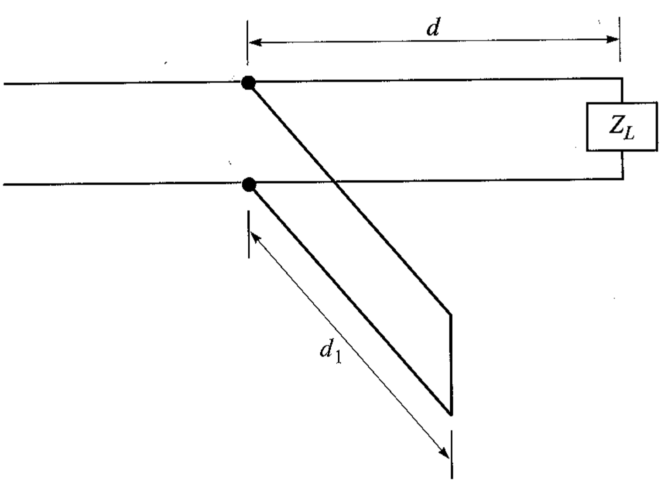


FIGURE 13.14
A short-circuited stub of length d_l, located a distance d from a load Z_L is used to provide a matched load to the left of the stub.

tant input impedance must be $1 + j0$. Since it is much easier to combine admittances in parallel than impedances, let us rephrase our goal in admittance language: the input admittance of the length d containing the load must be $1 + jb_{in}$ for the addition of the input admittance of the stub jb_{stub} to produce a total admittance of $1 + j0$. Hence the stub admittance is $-jb_{in}$. We shall therefore use the Smith chart as an admittance chart instead of an impedance chart.

The impedance of the load is $2.1 + j0.8$, and its location is at -11.5 cm. The admittance of the load is therefore $1/(2.1 + j0.8)$, and this value may be determined by adding one-quarter wavelength on the Smith chart, since Z_{in} for a quarter-wavelength line is R_0^2/Z_L, or $z_{in} = 1/z_L$, or $y_{in} = z_L$. Entering the chart (Fig. 13.15) at $z_L = 2.1 + j0.8$, we read 0.220 on the wtg scale; we add (or subtract) 0.250 and find the admittance $0.41 - j0.16$ corresponding to this impedance. This point is still located on the $s = 2.5$ circle. Now, at what point or points on this circle is the real part of the admittance equal to unity? There are two answers, $1 + j0.95$ at wtg $= 0.16$, and $1 - j0.95$ at wtg $= 0.34$, as shown in Fig. 13.15. Let us select the former value since this leads to the shorter stub. Hence $y_{stub} = -j0.95$, and the stub location corresponds to wtg $= 0.16$. Since the load admittance was found at wtg $= 0.470$, then we must move $(0.5 - 0.47) + 0.16 = 0.19$ wavelength to get to the stub location.

Finally, we may use the chart to determine the necessary length of the short-circuited stub. The input conductance is zero for any length of short-circuited stub, so we are restricted to the perimeter of the chart. At the short circuit, $y = \infty$ and wtg $= 0.250$. We find that $b_{in} = -0.95$ is achieved at wtg $= 0.379$, as shown in Fig. 13.15. The stub is therefore $0.379 - 0.250 = 0.129$ wavelength, or 9.67 cm long.

✔ **D13.8.** Standing wave measurements on a lossless 75-Ω line show maxima of 18 V and minima of 5 V. One minimum is located at a scale reading of 30 cm. With the load replaced by a short circuit, two adjacent minima are found at scale readings of 17 and 37 cm. Find: (a) s; (b) λ; (c) f; (d) Γ_L; (e) Z_L.

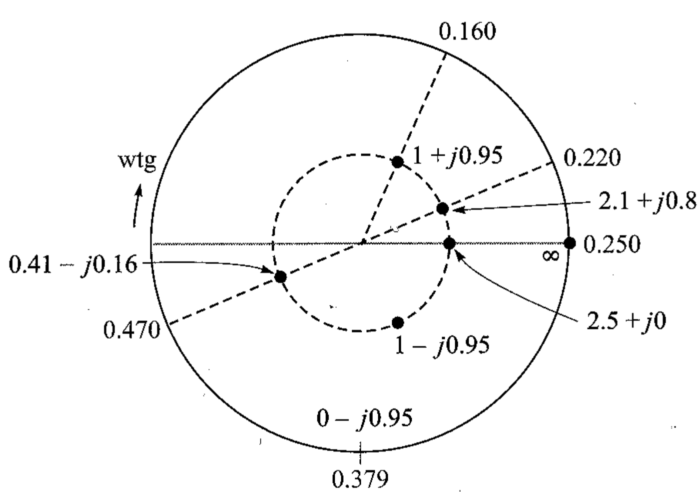


FIGURE 13.15
A normalized load $z_L = 2.1 + j0.8$ is matched by placing a 0.129-wavelength short-circuited stub 0.19 wavelength from the load.

Ans. 3.60; 0.400 m; 750 MHz; $0.704\angle{-33.0}$; $77.9 + j104.7\ \Omega$

✔ **D13.9.** A normalized load, $z_L = 2 - j1$, is located at $z = 0$ on a lossless 50-Ω line. Let the wavelength be 100 cm. (a) A short-circuited stub is to be located at $z = -d$. What is the shortest suitable value for d? (b) What is the shortest possible length of the stub? Find s: (c) on the main line for $z < -d$; (d) on the main line for $-d < z < 0$; (e) on the stub.

Ans. 12.5 cm; 12.5 cm; 1.00; 2.62; ∞.

13.6 TRANSIENTS ON TRANSMISSION LINES

Throughout this chapter, we have considered the operation of transmission lines under steady state conditions, in which voltage and current were sinusoidal and at a single frequency. In this section we move away from the simple time-harmonic case and consider transmission line responses to voltage step functions and pulses, grouped under the general heading of *transients*. Line operation in transient mode is important to study, as it allows us to understand how lines can be used to store and release energy (in pulse-forming applications, for example). Pulse propagation is important in general since digital signals, composed of sequences of pulses, are widely used.

We will confine our discussion to the propagation of transients in lines that are lossless and have no dispersion, so that the basic behavior and analysis methods may be learned. We must remember, however, that transient signals are necessarily composed of numerous frequencies, as Fourier analysis will show. Consequently, the question of dispersion in the line arises, since, as we have found, line propagation constants and reflection coefficients at complex loads will be frequency-dependent. So in general, pulses are likely to broaden with propagation distance, and pulse shapes may change when reflecting from a complex load. These issues will not be considered in detail here, but are readily addressed when the precise frequency dependences of β and Γ are known. In particular, $\beta(\omega)$ can be found by evaluating the imaginary part of γ, as given in Eq. (4), which would in general include the frequency dependences of R, C, G, and L arising from various mechanisms. For example, the skin effect (which affects both the conductor resistance and the internal inductance) will result in frequency-dependent R and L. Once $\beta(\omega)$ is known, pulse broadening can be evaluated using the methods presented in Chapter 12.

We begin our basic discussion of transients by considering a lossless transmission line of length, l, terminated by a matched load, $R_L = Z_0$, as shown in Fig. 13.16a. At the front end of the line is a battery of voltage, V_0, which is connected to the line by closing a switch. At time $t = 0$, the switch is closed, and the line voltage at $z = 0$ becomes equal to the battery voltage. This voltage, however, does not appear across the load until adequate time has elapsed for the propagation delay. Specifically, at $t = 0$, a voltage wave is initiated in the line

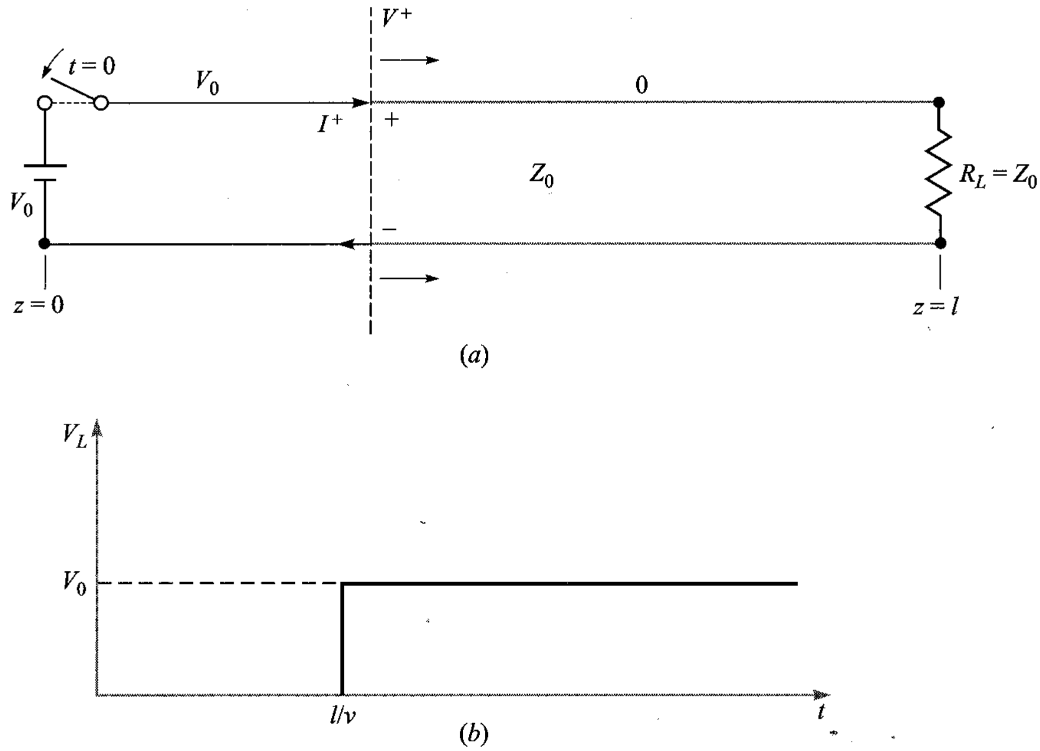


FIGURE 13.16
(*a*) Closing the switch at time $t = 0$ initiates voltage and current waves, V^+ and I^+. The leading edge of both waves is indicated by the dashed line, which propagates in the lossless line toward the load at velocity v. In this case, $V^+ = V_0$; the line voltage is V^+ everywhere to the left of the leading edge, where current is $I^+ = V^+/Z_0$. To the right of the leading edge, voltage and current are both zero. Clockwise current, indicated here, is treated as positive, and will occur when V^+ is positive. (*b*) Voltage across the load resistor as a function of time, showing the one-way transit time delay (l/v).

at the battery end, which then propagates toward the load. The leading edge of the wave, labeled V^+ in the figure, is of value $V^+ = V_0$. It can be thought of as a propagating step function, since at all points to the left of V^+, the line voltage is V_0; at all points to the right (not yet reached by the leading edge), the line voltage is zero. The wave propagates at velocity v, which in general is the group velocity in the line.[5] The wave reaches the load at time $t = l/v$, and then does not reflect, since the load is matched. The transient phase is thus over, and the load voltage is equal to the battery voltage. A plot of load voltage as a function of time is shown in Fig. 13.16*b*, indicating the propagation delay of $t = l/v$.

[5] Since we have a step function (composed of many frequencies) as opposed to a sinusoid at a single frequency, the wave will propagate at the group velocity. In a lossless line with no dispersion as considered in this section, $\beta = \omega\sqrt{LC}$, where L and C are constant with frequency. In this case we would find that the group and phase velocities are equal; (i.e, $d\omega/d\beta = \omega/\beta = v = 1/\sqrt{LC}$). We will thus write the velocity as v, knowing it to be both v_p and v_g.

Associated with the voltage wave, V^+, is a current wave whose leading edge is of value I^+. This wave is a propagating step function as well, whose value at all points to the left of V^+ is $I^+ = V^+/Z_0$; at all points to the right, current is zero. A plot of current through the load as a function of time will thus be identical in form to the voltage plot of Fig. 13.16*b*, except that the load current at $t = l/v$ will be $I_L = V^+/Z_0 = V_0/R_L$.

We next consider a more general case, in which the load of Fig. 13.16*a* is again a resistor, but is *not matched* to the line ($R_L \neq Z_0$). Reflections will occur at the load, thus complicating the problem. At $t = 0$, the switch is closed as before and a voltage wave, $V_1^+ = V_0$, propagates to the right. Upon reaching the load, however, the wave will now reflect, producing a back-propagating wave, V_1^-. The relation between V_1^- and V_1^+ is through the reflection coefficient at the load:

$$\frac{V_1^-}{V_1^+} = \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{46}$$

As V_1^- propagates back toward the battery, it leaves behind its leading edge a total voltage of $V_1^+ + V_1^-$. Voltage V_1^+ exists everywhere ahead of the V_1^- wave until it reaches the battery, whereupon the entire line now is charged to voltage $V_1^+ + V_1^-$. At the battery, the V_1^- wave reflects to produce a new forward wave, V_2^+. The ratio of V_2^+ and V_1^- is found through the reflection coefficient at the battery:

$$\frac{V_2^+}{V_1^-} = \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \tag{47}$$

where the impedance at the generator end, Z_g, is that of the battery, or zero.

V_2^+ (equal to $-V_1^-$) now propagates to the load, where it reflects to produce backward wave $V_2^- = \Gamma_L V_2^+$. This wave then returns to the battery, where it reflects with $\Gamma_g = -1$, and the process repeats. Note that with each round trip the wave voltage is reduced in magnitude since $|\Gamma_L| < 1$. Because of this the propagating wave voltages will eventually approach zero, and steady state is reached.

The voltage across the load resistor can be found at any given time by summing the voltage waves that have reached the load and have reflected from it up to that time. After many round trips, the load voltage will be in general:

$$\begin{aligned} V_L &= V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \dots \\ &= V_1^+\left(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 + \Gamma_g^2\Gamma_L^3 + \dots\right) \end{aligned}$$

Performing a simple factoring operation, the above becomes

$$V_L = V_1^+(1 + \Gamma_L)\left(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \dots\right) \tag{48}$$

Allowing time to approach infinity, the second term in parenthesis in (48) becomes the power series expansion for the expression $1/(1 - \Gamma_g \Gamma_L)$. Thus, in steady state we obtain,

$$V_L = V_1^+ \left(\frac{1 + \Gamma_L}{1 - \Gamma_g \Gamma_L} \right) \tag{49}$$

In our present example, $V_1^+ = V_0$ and $\Gamma_g = -1$. Substituting these into (49), we find the expected result in steady state: $V_L = V_0$.

A more general situation would involve a non zero impedance at the battery location, as shown in Fig. 13.17. In this case, a resistor of value R_g is positioned in series with the battery. When the switch is closed, the battery voltage appears across the series combination of R_g and the line characteristic impedance, Z_0. The value of the initial voltage wave, V_1^+, is thus found through simple voltage division, or

$$V_1^+ = \frac{V_0 Z_0}{R_g + Z_0} \tag{50}$$

With this initial value, the sequence of reflections and the development of the voltage across the load occurs in the same manner as determined by (48), with the steady state value determined by (49). The value of the reflection coefficient at the generator end, determined by (47), is $\Gamma_g = (R_g - Z_0)/(R_g + Z_0)$.

A useful way of keeping track of the voltage at any point in the line is through a *voltage reflection diagram*. Such a diagram for the line of Fig. 13.17 is shown in Fig. 13.18*a*. It is a two-dimensional plot in which position on the line, z, is shown on the horizontal axis. Time is plotted on the vertical axis, and is conveniently expressed as it relates to position and velocity through $t = z/v$. A vertical line, located at $z = l$, is drawn which, together with the ordinate, define the z axis boundaries of the transmission line. With the switch located at the

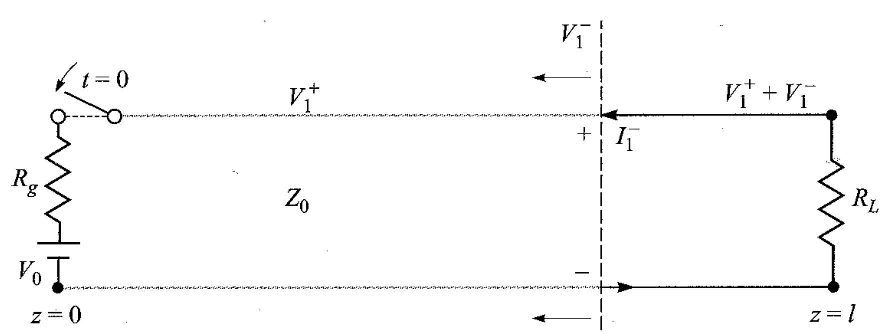


FIGURE 13.17
With a series resistance at the battery location, voltage division occurs when the switch is closed, such that $V_0 = V_{Rg} + V_1^+$. Shown is the first reflected wave, which leaves voltage $V_1^+ + V_1^-$ behind its leading edge. Associated with the wave is current I_1^-, which is $-V_1^-/Z_0$. Counter-clockwise current is treated as negative, and will occur when V_1^- is positive.

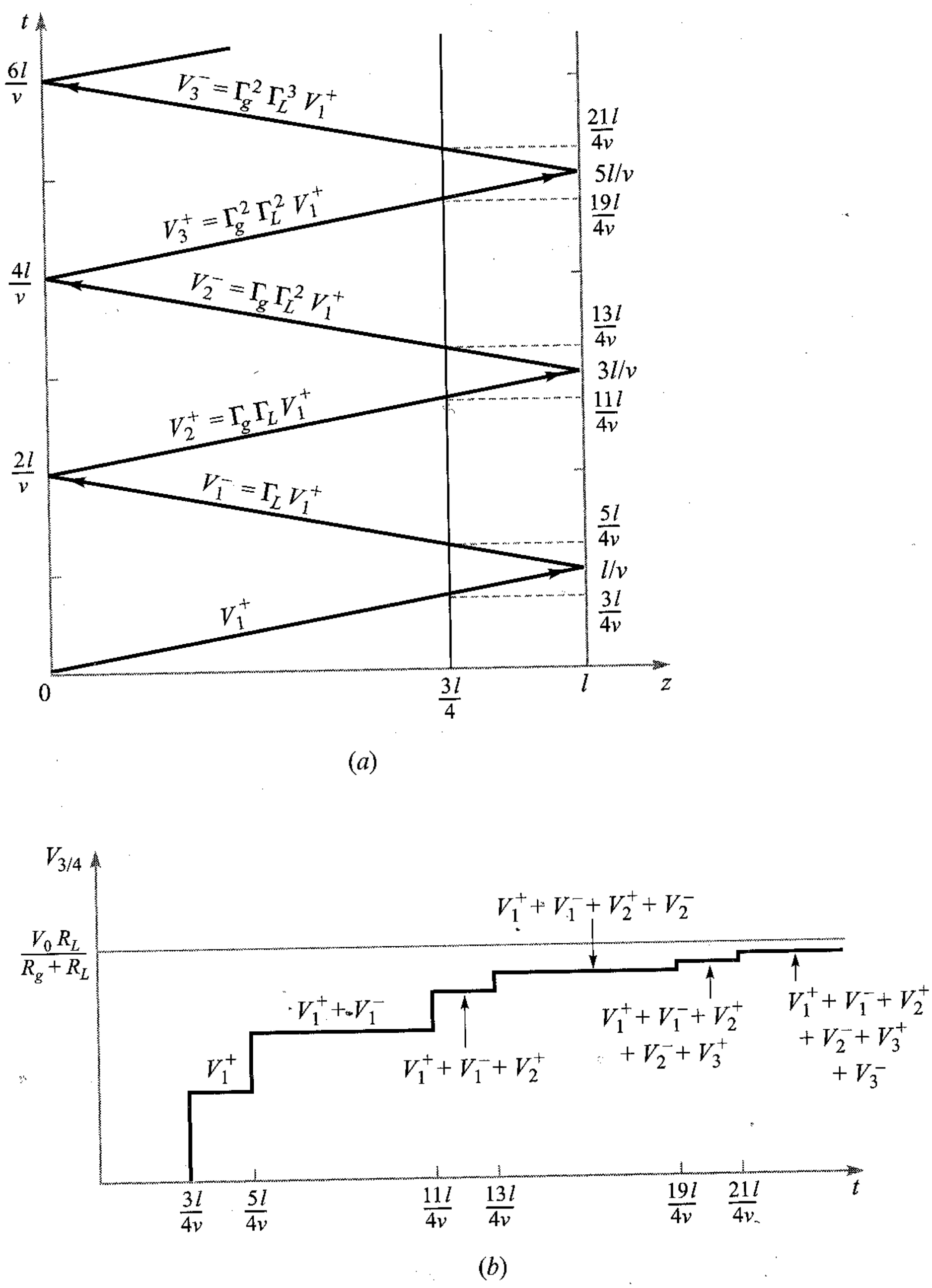


FIGURE 13.18
(*a*) Voltage reflection diagram for the line of Fig. 13.17. A reference line, drawn at $z = 3l/4$, is used to evaluate the voltage at that position as a function of time. (*b*) The line voltage at $z = 3l/4$ as determined from the reflection diagram of (*a*). Note that the voltage approaches the expected $V_0 R_L/(R_g + R_L)$ as time approaches infinity.

battery position, the initial voltage wave, V_1^+, starts at the origin, or lower left corner of the diagram ($z = t = 0$). The location of the leading edge of V_1^+ as a function of time is shown as the diagonal line that joins the origin to the point along the right-hand vertical line that corresponds to time $t = l/v$ (the one-way transit time). From there (the load location) the position of the leading edge of the reflected wave, V_1^-, is shown as a "reflected" line which joins the $t = l/v$ point on the right boundary to the $t = 2l/v$ point on the ordinate. From there (at the battery location) the wave reflects again, forming V_2^+, shown as a line parallel to that for V_1^+. Subsequent reflected waves are shown, and their values are labeled.

The voltage as a function of time at a given position in the line can now be determined by adding the voltages in the waves as they intersect a vertical line, drawn at the desired location. This addition is performed starting at the bottom of the diagram ($t = 0$) and progressing upward (in time). Whenever a voltage wave crosses the vertical line, its value is added to the total at that time. For example, the voltage at a location three-fourths the distance from the battery to the load is plotted in Fig. 13.18*b*. To obtain this plot, the line $z = (3/4)l$ is drawn on the diagram. Whenever a wave crosses this line, the voltage in the wave is added to the voltage that has accumulated at $z = (3/4)l$ over all earlier times. This general procedure enables one to easily determine the voltage at any specific time and location. In doing so, the terms in (48) that have occurred up to the chosen time are being added, but with information on the time at which each term appears.

Line current can be found in a similar way through a *current reflection diagram*. It is easiest to construct the current diagram directly from the voltage diagram by determining a value for current that is associated with each voltage wave. In dealing with current, it is important to keep track of the *sign* of the current as it relates to the voltage waves and their polarities. Referring to Figs. 13.16*a* and 13.17, we use the convention in which current associated with a *forward-z* traveling voltage wave of positive polarity is positive. This would result in current that flows in the clockwise direction, as shown in the Fig. 13.16*a*. Current associated with a *backward-z* traveling voltage wave of positive polarity (thus flowing counterclockwise) is negative. Such a case is illustrated in Fig. 13.17. In our two-dimensional transmission line drawings, we assign positive polarity to voltage waves propagating in *either* direction if the upper conductor carries a positive charge and the lower conductor a negative charge. In Figs. 13.16*a* and 13.17, both voltage waves are of positive polarity, so their associated currents will be net positive for the forward wave, and net negative for the backward wave. In general, we write

$$I^+ = \frac{V^+}{Z_0} \tag{51}$$

and

$$I^- = -\frac{V^-}{Z_0} \tag{52}$$

Finding the current associated with a backward-propagating voltage wave immediately requires a minus sign, as (52) indicates.

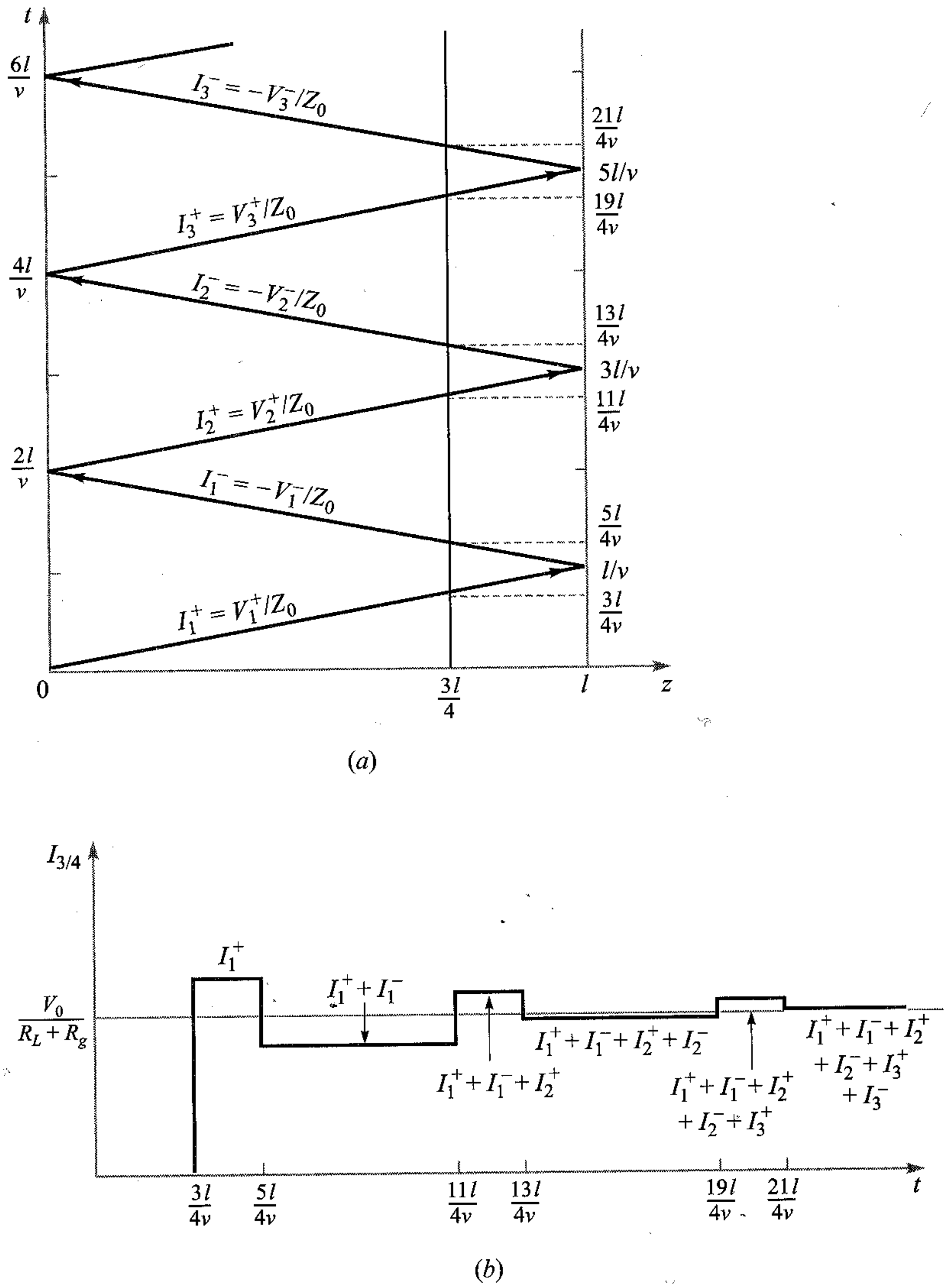


FIGURE 13.19
(a) Current reflection diagram for the line of Fig. 13.17 as obtained from the voltage diagram of Fig. 13.18a. (b) Current at the $z = 3l/4$ position as determined from the current reflection diagram, showing the expected steady state value of $V_0/(R_L + R_g)$.

Fig. 13.19a shows the current reflection diagram that is derived from the voltage diagram of Fig. 13.18a. Note that the current values are labeled in terms of the voltage values, with the appropriate sign added as per (51) and (52). Once the current diagram is constructed, current at a given location and time can be found in exactly the same manner as voltage is found using the voltage diagram.

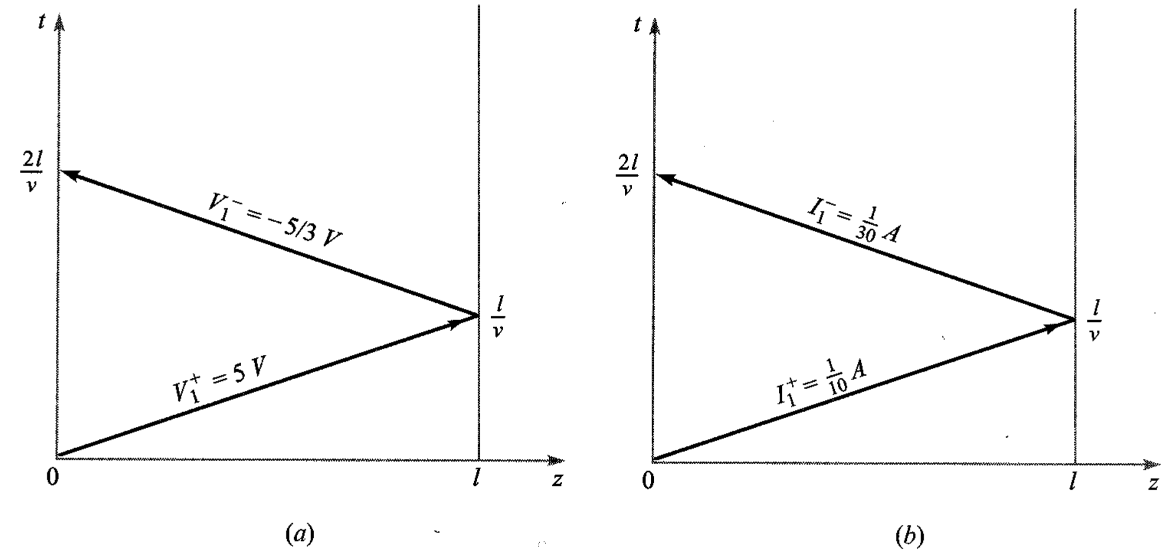


FIGURE 13.20
Voltage (*a*) and current (*b*) reflection diagrams for Example 13.5.

Fig. 13.19*b* shows the current as a function of time at the $z = (3/4)l$ position, determined by summing the current wave values as they cross the vertical line drawn at that location.

Example 13.5

In the line shown in Fig. 13.17, $R_g = Z_0 = 50\,\Omega$, $R_L = 25\,\Omega$, and the battery voltage is $V_0 = 10$ V. The switch is closed at time $t = 0$. Determine the voltage at the load resistor and the current in the battery as functions of time.

Solution. Voltage and current reflection diagrams are shown in Fig. 13.20*a* and *b*. At the moment the switch is closed, half the battery voltage appears across the 50 ohm resistor, with the other half comprising the initial voltage wave. Thus $V_1^+ = (1/2)V_0 = 5$ V. The wave reaches the 25 ohm load, where it reflects with reflection coefficient

$$\Gamma_L = \frac{25 - 50}{25 + 50} = -\frac{1}{3}$$

So $V_1^- = -(1/3)V_1^+ = -5/3$ V. This wave returns to the battery, where it encounters reflection coefficient, $\Gamma_g = 0$. Thus, no further waves appear; steady state is reached.

Once the voltage wave values are known, the current reflection diagram can be constructed. The values for the two current waves are

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{5}{50} = \frac{1}{10} \text{ A}$$

and

$$I_1^- = -\frac{V_1^-}{Z_0} = -\left(-\frac{5}{3}\right)\left(\frac{1}{50}\right) = \frac{1}{30}\ \text{A}$$

Note that no attempt is made here to derive I_1^- from I_1^+. They are both obtained independently from their respective voltages.

The voltage at the load as a function of time is now found by summing the voltages along the vertical line at the load position. The resulting plot is shown in Fig. 13.21*a*. Current in the battery is found by summing the currents along the vertical axis, with the resulting plot shown as Fig. 13.21*b*. Note that in steady state, we treat the circuit as lumped, with the battery in series with the 50 and 25 ohm resistors. Therefore, we expect to see a steady-state current through the battery (and everywhere else) of

$$I_B(\text{steady state}) = \frac{10}{50+25} = \frac{1}{7.5}\ \text{A}$$

This value is also found from the current reflection diagram for $t > 2l/v$. Similarly, the steady-state load voltage should be

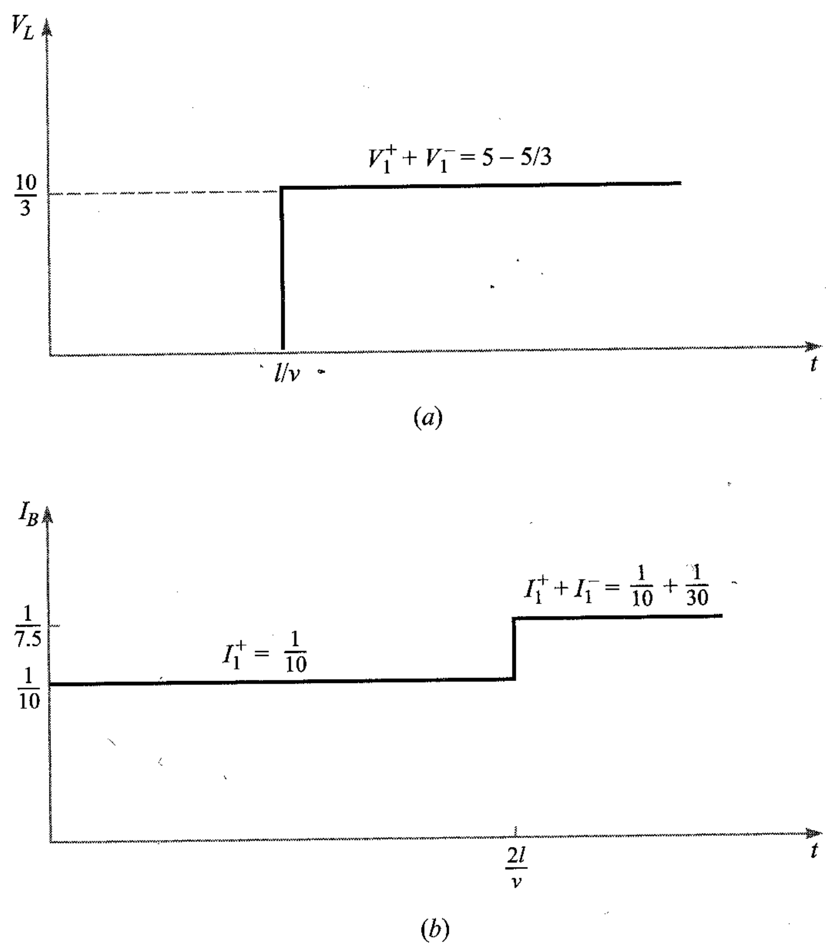


FIGURE 13.21
Voltage across the load (*a*), and current in the battery (*b*), as determined from the reflection diagrams of Fig. 13.20 (Example 13.5).

$$V_L(\text{steady state}) = V_0 \frac{R_L}{R_g + R_L} = \frac{(10)(25)}{50 + 25} = \frac{10}{3} \text{ V}$$

which is found also from the voltage reflection diagram for $t > l/v$.

Another type of transient problem involves lines that are *initially charged.* In these cases, the manner in which the line discharges through a load is of interest. Consider the situation shown in Fig. 13.22, in which a charged line of characteristic impedance Z_0 is discharged through a resistor of value R_g when a switch at the resistor location is closed.[6] We consider the resistor at the $z = 0$ location; the other end of the line is open (as would be necessary) and is located at $z = l$.

When the switch is closed, current I_R begins to flow through the resistor, and the line discharge process begins. This current does not immediately flow everywhere in the transmission line, but begins at the resistor, and establishes its presence at more distant parts of the line as time progresses. By analogy, consider a long line of automobiles at a red light. When the light turns green, the cars at the front move through the intersection first, followed successively by those further toward the rear. The point which divides cars in motion and those standing still is in fact a wave which propagates toward the back of the line. In the transmission line, the flow of charge progresses in a similar way. A voltage wave, V_1^+, is initiated and propagates to the right. To the left of its leading edge, charge is in motion; to the right of the leading edge, charge is stationary, and carries its original density. Accompanying the charge in motion to the left of V_1^+ is a drop in the charge density as the discharge process occurs, and so the line voltage to the left of V_1^+ is partially reduced. This voltage will be given by the sum of the initial voltage, V_0, and V_1^+, which means that V_1^+ must in fact be

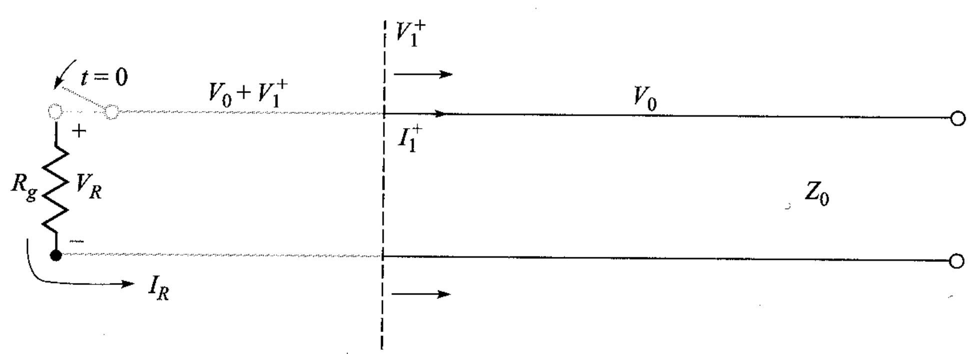


FIGURE 13.22
In an initially charged line, closing the switch as shown initiates a voltage wave of opposite polarity to that of the initial voltage. The wave thus depletes the line voltage and will fully discharge the line in one round trip if $R_g = Z_0$.

---

[6] Even though this is a load resistor, we will call it R_g since it is located at the front (generator) end of the line.

negative (or of opposite sign to V_0). The line discharge process is analyzed by keeping track of V_1^+ as it propagates and undergoes multiple reflections at the two ends. Voltage and current reflection diagrams are used for this purpose in much the same way as before.

Referring to Fig. 13.22, we see that for positive V_0 the current flowing through the resistor will be counterclockwise, and hence negative. We also know that continuity requires that the resistor current be equal to the current associated with the voltage wave, or

$$I_R = -I_1^+ = -\frac{V_1^+}{Z_0}$$

Now the resistor voltage will be

$$V_R = V_0 + V_1^+ = I_R R_g = -I_1^+ R_g = -\frac{V_1^+}{Z_0} R_g$$

We solve for V_1^+ to obtain

$$V_1^+ = \frac{-V_0 Z_0}{Z_0 + R_g} \tag{53}$$

Having found V_1^+, we can set up the voltage and current reflection diagrams. That for voltage is shown in Fig. 13.23. Note that the initial condition of voltage V_0 everywhere on the line is accounted for by assigning voltage V_0 to the horizontal axis of the voltage diagram. The diagram is otherwise drawn as before, but with $\Gamma_L = 1$ (at the open-circuited load end). Variations in how the line discharges thus depend on the resistor value at the switch end, R_g, which determines the reflection coefficient, Γ_g, at that location. The current reflection diagram is derived from the voltage diagram in the usual way. There is no initial current to consider.

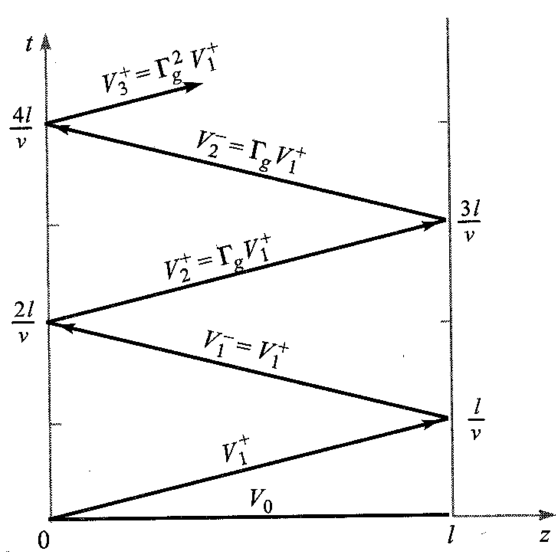


FIGURE 13.23
Voltage reflection diagram for the charged line of Fig. 13.22, showing the initial condition of V_0 everywhere on the line at $t = 0$.

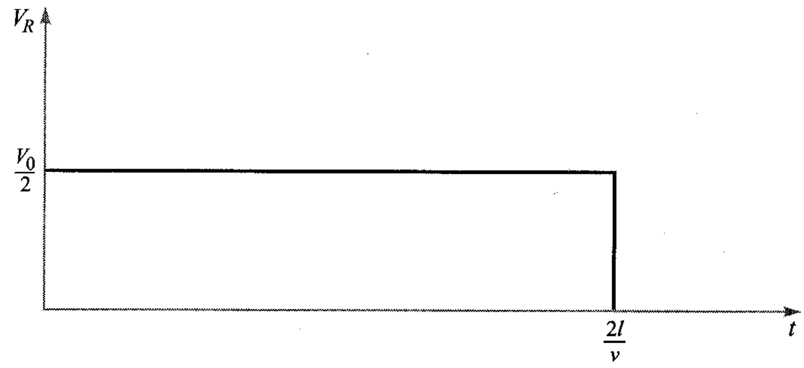


FIGURE 13.24
Voltage across the resistor as a function of time, as determined from the reflection diagram of Fig. 13.23, in which $R_g = Z_0$ ($\Gamma_g = 0$).

A special case of practical importance is that in which the resistor is matched to the line, or $R_g = Z_0$. In this case, Eq. (53) gives $V_1^+ = -V_0/2$. The line fully discharges in one round-trip of V_1^+, and produces a voltage across the resistor of value $V_R = V_0/2$, which persists for time $T = 2l/v$. The resistor voltage as a function of time is shown in Fig. 13.24. The transmission line in this application is known as a *pulse-forming line*. Pulses that are generated in this way are well-formed and of low noise, provided the switch is sufficiently fast. Commercial units are available that are capable of generating high-voltage pulses of widths on the order of a few nanoseconds, using thyratron-based switches.

When the resistor is not matched to the line, full discharge still occurs, but does so over several reflections, leading to a complicated pulse shape.

Example 13.6

In the charged line of Fig. 13.22, the characteristic impedance is $Z_0 = 100\ \Omega$, and $R_g = 100/3\ \Omega$. The line is charged to an initial voltage, $V_0 = 160$ V, and the switch is closed at time $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/v$ (four round-trips).

Solution. With the given values of R_g and Z_0, Eq. (47) gives $\Gamma_g = -1/2$. Then, with $\Gamma_L = 1$, and using (53), we find

$$V_1^+ = V_1^- = -\frac{3}{4}V_0 = -120\,\text{V}$$
$$V_2^+ = V_2^- = \Gamma_g V_1^- = +60\,\text{V}$$
$$V_3^+ = V_3^- = \Gamma_g V_2^- = -30\,\text{V}$$
$$V_4^+ = V_4^- = \Gamma_g V_3^- = +15\,\text{V}$$

Using these values on the voltage reflection diagram, we evaluate the voltage in time at the resistor location by moving up the left-hand vertical axis, adding voltages as we progress, and beginning with $V_0 + V_1^+$ at $t = 0$. Note that when we add voltages along the vertical axis, we are encountering the intersection points between incident and reflected waves, which occur (in time) at each integer multiple of $2l/v$. So, when moving up the axis, we add the voltages of *both* waves to our total at each occurrence. The voltage within each time interval is thus:

$$
\begin{aligned}
V_R &= V_0 + V_1^+ = 40\,\text{V} && (0 < t < 2l/v)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ = -20\,\text{V} && (2l/v < t < 4l/v)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ = 10\,\text{V} && (4l/v < t < 6l/v)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + V_4^+ = -5\,\text{V} && (6l/v < t < 8l/v)
\end{aligned}
$$

The resulting voltage plot over the desired time range is shown in Fig. 13.25*a*.

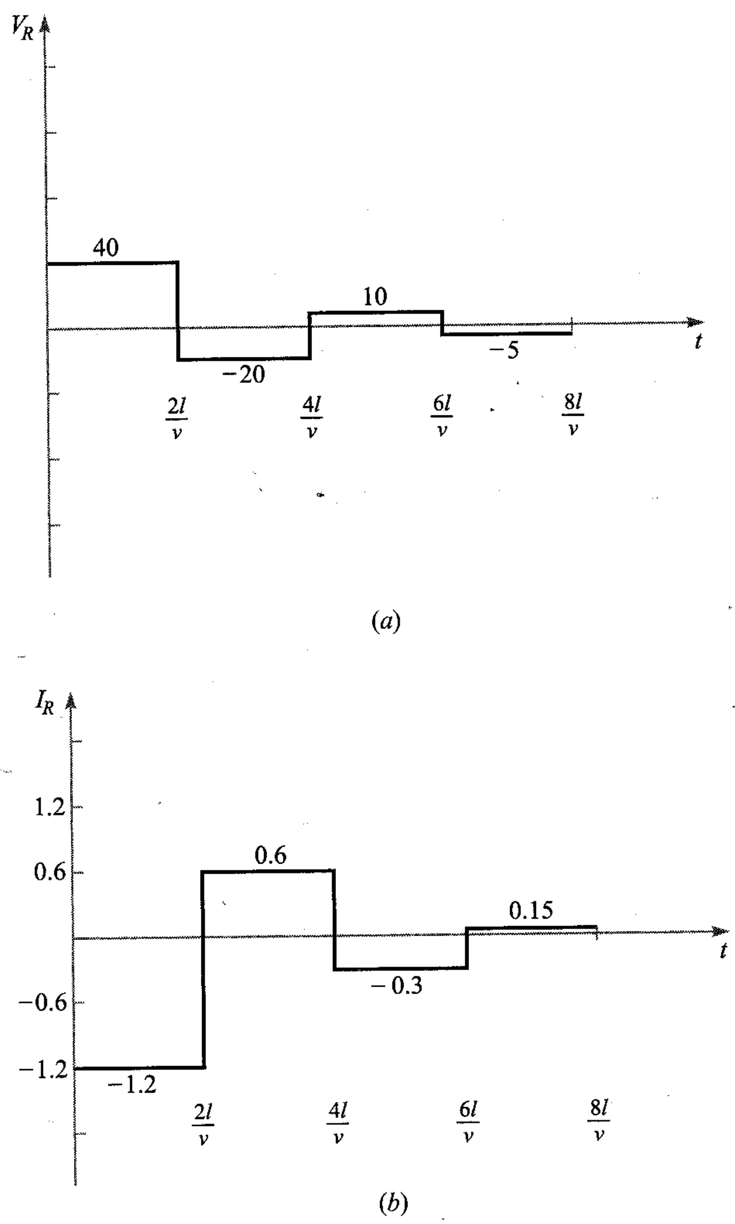


FIGURE 13.25
Resistor voltage (*a*) and current (*b*) as functions of time for the line of Fig. 13.22, with values as specified in Example 13.6.

The current through the resistor is most easily obtained by dividing the voltages in Fig. 13.25*a* by $-R_g$. As a demonstration, we can also use the current diagram of Fig. 13.19*a* to obtain this result. Using (51) and (52), we evaluate the current waves as follows:

$$I_1^+ = V_1^+/Z_0 = -1.2\,\mathrm{A}$$
$$I_1^- = -V_1^-/Z_0 = +1.2\,\mathrm{A}$$
$$I_2^+ = -I_2^- = V_2^+/Z_0 = +0.6\,\mathrm{A}$$
$$I_3^+ = -I_3^- = V_3^+/Z_0 = -0.30\,\mathrm{A}$$
$$I_4^+ = -I_4^- = V_4^+/Z_0 = +0.15\,\mathrm{A}$$

Using the above values on the current reflection diagram, Fig. 13.19*a*, we add up currents in the resistor in time by moving up the left-hand axis, as we did with the voltage diagram. The result is shown in Fig. 13.25*b*. As a further check to the correctness of our diagram construction, we note that current at the open end of the line ($Z = l$) must always be zero. Therefore, summing currents up the right-hand axis must give a zero result for all time. The reader is encouraged to verify this.

SUGGESTED REFERENCES

1. Brown, R. G., R. A. Sharpe, W. L. Hughes, and R. E. Post: "Lines, Waves, and Antennas," 2d ed., The Ronald Press Company, New York, 1973. Transmission lines are covered in the first six chapters, with numerous examples.
2. Cheng, D. K.: "Field and Wave Electromagnetics," 2nd ed., Addison-Wesley Publishing Company, Reading, Mass., 1989. Provides numerous examples of Smith Chart problems and transients.
3. Seshadri, S. R.: "Fundamentals of Transmission Lines and Electromagnetic Fields," Addison-Wesley Publishing Company, Reading, Mass., 1971.
4. Weeks, W. L.: "Transmission and Distribution of Electrical Energy," Harper and Row, Publishers, New York, 1981. Line parameters for various configurations of power transmission and distribution systems are discussed in Chap. 2, along with typical parameter values.

PROBLEMS

13.1 The parameters of a certain transmission line operating at 6×10^8 rad/s are $L = 0.4\,\mu$H/m, $C = 40$ pF/m, $G = 80$ mS/m, and $R = 20\ \Omega$/m. (a) Find γ, α, β, λ, and Z_0. (b) If a voltage wave travels 20 m down the line, by what percentage is its amplitude reduced, and by how many degrees is its phase shifted?

13.2 A lossless transmission line with $Z_0 = 60\,\Omega$ is being operated at 60 MHz. The velocity on the line is 3×10^8 m/s. If the line is short-circuited at $z = 0$, find Z_{in} at $z =$: (a) -1 m; (b) -2 m; (c) -2.5 m; (d) -1.25 m.

13.3 The characteristic impedance of a certain lossless transmission line is $72\,\Omega$. If $L = 0.5\,\mu\text{H/m}$, find: (a) C; (b) v_p; (c) β if $f = 80$ MHz. (d) The line is terminated with a load of 60 Ω. Find Γ and s.

13.4 A lossless transmission line having $Z_0 = 120\ \Omega$ is operating at $\omega = 5 \times 10^8$ rad/s. If the velocity on the line is 2.4×10^8 m/s, find: (a) L; (b) C. (c) Let Z_L be represented by an inductance of 0.6 μH in series with a 100-Ω resistance. Find Γ and s.

13.5 Two characteristics of a certain lossless transmission line are $Z_0 = 50\,\Omega$ and $\gamma = 0 + j0.2\pi\,\text{m}^{-1}$ at $f = 60$ MHz: (a) find L and C for the line. (b) A load $Z_L = 60 + j80\ \Omega$ is located at $z = 0$. What is the shortest distance from the load to a point at which $Z_{in} = R_{in} + j0$?

13.6 The propagation constant of a lossy transmission line is $1 + j2\,\text{m}^{-1}$, and its characteristic impedance is $20 + j0\ \Omega$ at $\omega = 1$ Mrad/s. Find L, C, R, and G for the line.

13.7 The dimensions of the outer conductor of a coaxial cable are b and c, $c > b$. Assume $\sigma = \sigma_c$ and let $\mu = \mu_0$. Find the magnetic energy stored per unit length in the region $b < r < c$ for a uniformly distributed total current I flowing in the opposite directions in the inner and outer conductors.

13.8 The conductors of a coaxial transmission line are copper ($\sigma_c = 5.8 \times 10^7$ S/m), and the dielectric is polyethylene ($\epsilon'_R = 2.26$, $\sigma/\omega\epsilon' = 0.0002$). If the inner radius of the outer conductor is 4 mm, find the radius of the inner conductor so that: (a) $Z_0 = 50\ \Omega$; (b) $C = 100$ pF/m; (c) $L = 0.2\,\mu\text{H/m}$. A lossless line can be assumed.

13.9 Two aluminum-clad steel conductors are used to construct a two-wire transmission line. Let $\sigma_{\text{Al}} = 3.8 \times 10^7$ S/m, $\sigma_{\text{St}} = 5 \times 10^6$ S/m, and $\mu_{\text{St}} = 100\,\mu\text{H/m}$. The radius of the steel wire is 0.5 in., and the aluminum coating is 0.05 in thick. The dielectric is air, and the center-to-center wire separation is 4 in. Find C, L, G, and R for the line at 10 MHz.

13.10 Each conductor of a two-wire transmission line has a radius of 0.5 mm; their center-to-center separation is 0.8 cm. Let $f = 150$ MHz, and assume σ and σ_c are zero. Find the dielectric constant of the insulating medium if: (a) $Z_0 = 300\ \Omega$; (b) $C = 20$ pF/m; (c) $v_p = 2.6 \times 10^8$ m/s.

13.11 Pertinent dimensions for the transmission line shown in Fig. 13.4 are $b = 3$ mm and $d = 0.2$ mm. The conductors and the dielectric are nonmagnetic. (a) If the characteristic impedance of the line is 15 Ω, find ϵ'_R. Assume a low-loss dielectric. (b) Assume copper conductors and operation at 2×10^8 rad/s. If $RC = GL$, determine the loss tangent of the dielectric.

13.12 A transmission line constructed from perfect conductors and an air dielectric is to have a maximum dimension of 8 mm for its cross section. The line is to be used at high frequencies. Specify the dimensions if it is: (a) a two-wire line with $Z_0 = 300\ \Omega$; (b) a planar line with $Z_0 = 15\ \Omega$; (c) a 72-Ω coax having a zero-thickness outer conductor.

13.13 The incident voltage wave on a certain lossless transmission line for which $Z_0 = 50\ \Omega$ and $v_p = 2 \times 10^8$ m/s is $V^+(z, t) = 200\cos(\omega t - \pi z)$ V. (a) Find ω. (b) Find $I^+(z, t)$. The section of line for which $z > 0$ is replaced by a load $Z_L = 50 + j30\ \Omega$ at $z = 0$. Find: (c) Γ_L; (d) $V_s^-(z)$; (e) V_s at $z = -2.2$ m.

13.14 Coaxial lines 1 and 2 have the following parameters: $\mu_1 = \mu_2 = \mu_0$, $\sigma_1 = \sigma_2 = 0$, $\epsilon'_{R1} = 2.25$, $\epsilon'_{R2} = 4$, $a_1 = a_2 = 0.8$ mm, $b_1 = 6$ mm, $b_2 = 3$ mm, $Z_{L2} = Z_{02}$, and Z_{L1} is Z_{in2}. (a) Find Z_{01} and Z_{02}. (b) Find s on line 1. (c) If a 20-cm length of line 1 is inserted immediately in front of Z_{L2} and $f = 300$ MHz, find s on line 2.

13.15 For the transmission line represented in Fig. 13.26, find $V_{s,out}$ if $f =$:(a) 60 Hz; (b) 500 kHz.

13.16 A 300-Ω transmission line is 0.8 m long and terminated with a short circuit. The line is operating in air with a wavelength of 0.8 m and is lossless. (a) If the input voltage amplitude is 10 V, what is the maximum voltage amplitude at any point on the line? (b) What is the current amplitude in the short circuit?

13.17 Determine the average power absorbed by each resistor in Fig. 13.27.

13.18 The line shown in Fig. 13.28 is lossless. Find s on both sections 1 and 2.

13.19 A lossless transmission line is 50 cm in length and operating at a frequency of 100 MHz. The line parameters are $L = 0.2\ \mu$H/m and $C = 80$ pF/m. The line is terminated in a short circuit at $z = 0$, and there is a load $Z_L = 50 + j20\ \Omega$ across the line at location $z = -20$ cm. What average power is delivered to Z_L if the input voltage is $100\angle 0°$ V?

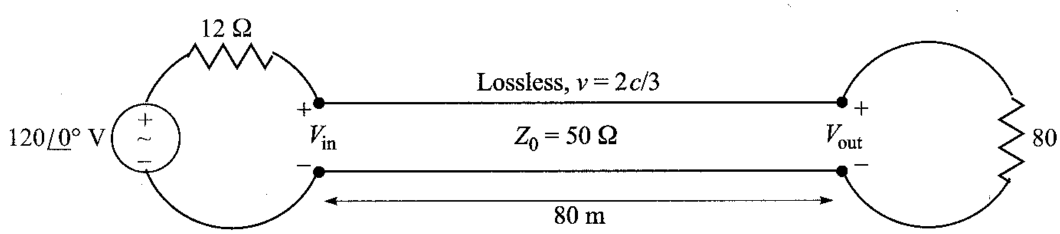


FIGURE 13.26
See Problem 15.

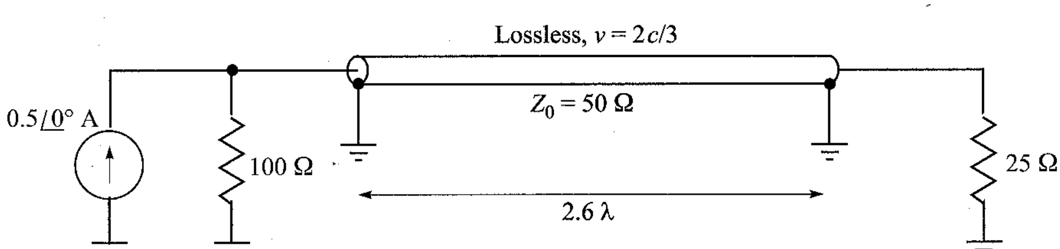


FIGURE 13.27
See Problem 17.

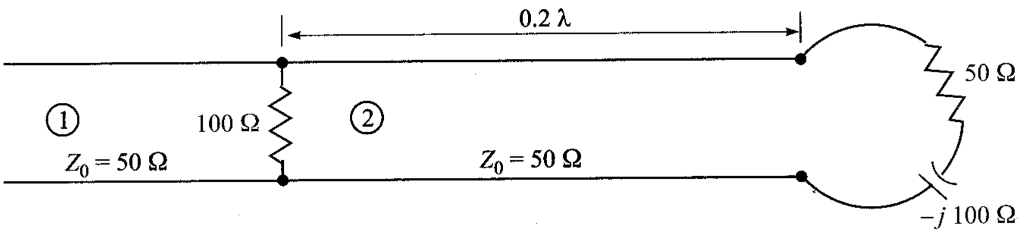


FIGURE 13.28
See Problem 18.

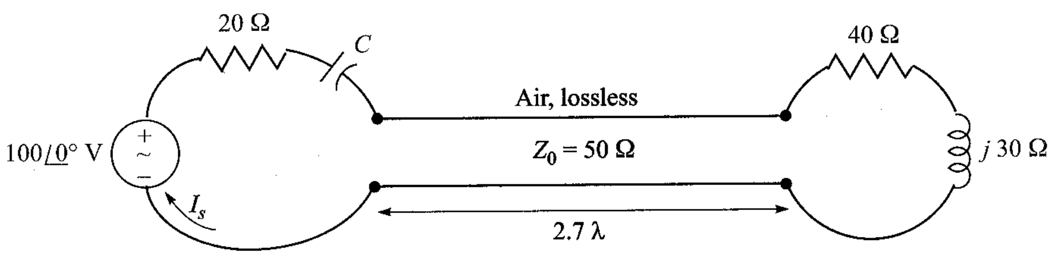


FIGURE 13.29
See Problem 20.

13.20 (a) Determine s on the transmission line of Fig. 13.29. Note that the dielectric is air. (b) Find the input impedance. (c) If $1/\omega C = 10\ \Omega$, find I_s. (d) What value of C will produce a maximum value for $|I_s|$ at $\omega = 1$ Grad/s? For this value of C, calculate the average power: (e) supplied by the source; (f) delivered to $Z_L = 40 + j30\ \Omega$.

13.21 A lossless line having an air dielectric has a characteristic impedance of 400 Ω. The line is operating at 200 MHz and $Z_{in} = 200 - j200\ \Omega$. Use analytic methods or the Smith chart (or both) to find: (a) s, (b) Z_L, if the line is 1 m long; (c) the distance from the load to the nearest voltage maximum.

13.22 A lossless two-wire line has a characteristic impedance of 300 Ω and a capacitance of 15 pF/m. The load at $z = 0$ consists of a 600-Ω resistor in parallel with a 10-pF capacitor. If $\omega = 10^8$ rad/s and the line is 20 m long, use the Smith chart to find: (a) $|\Gamma_L|$; (b) s; (c) Z_{in}.

13.23 The normalized load on a lossless transmission line is $2 + j1$. Let $l = 20$ m and make use of the Smith chart to find: (a) the shortest distance from the load to a point at which $z_{in} = r_{in} + j0$, where $r_{in} > 0$; (b) z_{in} at this point. (c) The line is cut at this point and the portion containing z_L is thrown away. A resistor $r = r_{in}$ of part (a) is connected across the line. What is s on the remainder of the line? (d) What is the shortest distance from this resistor to a point at which $z_{in} = 2 + j1$?

13.24 With the aid of the Smith chart, plot a curve of $|Z_{in}|$ vs. l for the transmission line shown in Fig. 13.30. Cover the range $0 < l/\lambda < 0.25$.

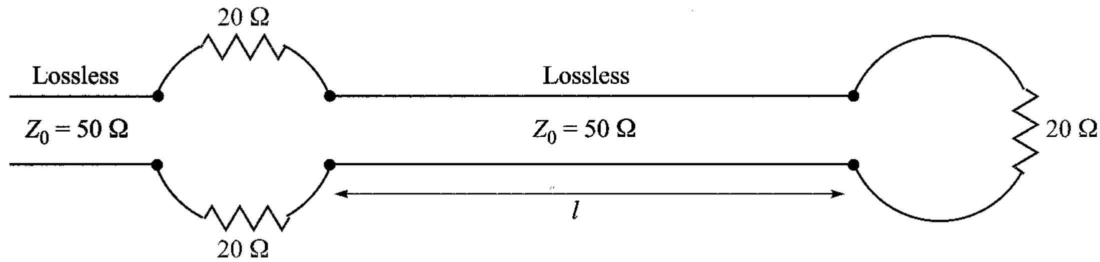


FIGURE 13.30
See Problem 24.

13.25 A 300-Ω transmission line is short-circuited at $z = 0$. A voltage maximum, $|V|_{max} = 10$ V, is found at $z = -25$ cm, and the minimum voltage, $|V|_{min} = 0$ is at $z = -50$ cm. Use the Smith chart to find Z_L (with the short circuit replaced by the load) if the voltage readings are: (a) $|V|_{max} = 12$ V at $z = -5$ cm, and $|V|_{min} = 5$ V; (b) $|V|_{max} = 17$ V at $z = -20$ cm, and $|V|_{min} = 0$.

13.26 A lossless 50-Ω transmission line operates with a velocity that is 3/4 c. A load $Z_L = 60 + j30\ \Omega$ is located at $z = 0$. Use the Smith chart to find: (a) s; (b) the distance from the load to the nearest voltage minimum if $f = 300$ MHz; (c) the input impedance if $f = 200$ MHz and the input is at $z = -110$ cm.

13.27 The characteristic admittance ($Y_0 = 1/Z_0$) of a lossless transmission line is 20 mS. The line is terminated in a load $Y_L = 40 - j20$ mS. Make use of the Smith chart to find: (a) s; (b) Y_{in} if $l = 0.15\lambda$; (c) the distance in wavelengths from Y_L to the nearest voltage maximum.

13.28 The wavelength on a certain lossless line is 10 cm. If the normalized input impedance is $z_{in} = 1 + j2$, use the Smith chart to determine: (a) s; (b) z_L, if the length of the line is 12 cm; (c) x_L, if $z_L = 2 + jx_L$ where $x_L > 0$.

13.29 A standing wave ratio of 2.5 exists on a lossless 60-Ω line. Probe measurements locate a voltage minimum on the line whose location is marked by a small scratch on the line. When the load is replaced by a short circuit, the minima are 25 cm apart, and one minimum is located at a point 7 cm toward the source from the scratch. Find Z_L.

13.30 A 2-wire line constructed of lossless wire of circular cross section is gradually flared into a coupling loop that looks like an egg beater. At the point X, indicated by the arrow in Fig. 13.31, a short circuit is placed across the line. A probe is moved along the line and indicates that the first voltage minimum to the left of X is 16 cm from X. With the short circuit removed, a voltage minimum is found 5 cm to the left of X, and a voltage maximum is located that is 3 times the voltage of the minimum. Use the Smith chart to determine: (a) f; (b) s; (c) the normalized input impedance of the egg beater as seen looking to the right at point X.

13.31 In order to compare the relative sharpness of the maxima and minima of a standing wave, assume a load $z_L = 4 + j0$ is located at $z = 0$. Let

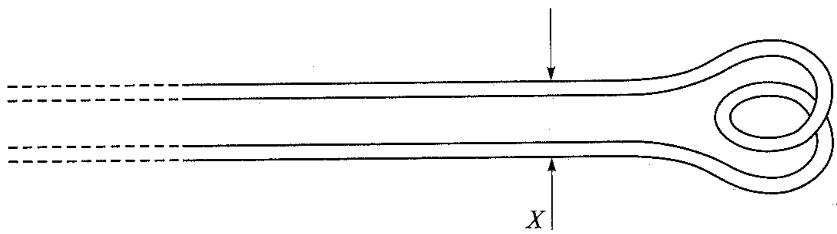


FIGURE 13.31
See Problem 30.

$|V|_{min} = 1$ and $\lambda = 1$ m. Determine the width of the: (a) minimum where $|V| < 1.1$; (b) maximum where $|V| > 4/1.1$.

13.32 A lossless line is operating with $Z_0 = 40\ \Omega$, $f = 20$ MHz, and $\beta = 7.5\pi$ rad/m. With a short circuit replacing the load, a minimum is found at a point on the line marked by a small spot of puce paint. With the load installed, it is found that $s = 1.5$ and a voltage minimum is located 1 m toward the source from the puce dot. (a) Find Z_L. (b) What load would produce $s = 1.5$ with $|V|_{max}$ at the paint spot?

13.33 In Fig. 13.14, let $Z_L = 40 - j10\ \Omega$, $Z_0 = 50\ \Omega$, $f = 800$ MHz, and $v = c$. (a) Find the shortest length d_1 of a short-circuited stub, and the shortest distance d that it may be located from the load to provide a perfect match on the main line to the left of the stub. (b) Repeat for an open-circuited stub.

13.34 The lossless line shown in Fig. 13.32 is operating with $\lambda = 100$ cm. If $d_1 = 10$ cm, $d = 25$ cm, and the line is matched to the left of the stub, what is Z_L?

13.35 A load, $Z_L = 25 + j75\ \Omega$, is located at $z = 0$ on a lossless two-wire line for which $Z_0 = 50\ \Omega$ and $v = c$. (a) If $f = 300$ MHz, find the shortest distance d ($z \doteq -d$) at which the input admittance has a real part equal to $1/Z_0$ and a negative imaginary part. (b) What value of capacitance C should be connected across the line at that point to provide unity standing wave ratio on the remaining portion of the line?

13.36 The two-wire lines shown in Fig. 13.33 are all lossless and have $Z_0 = 200\ \Omega$. Find d and the shortest possible value for d_1 to provide a matched load if $\lambda = 100$ cm.

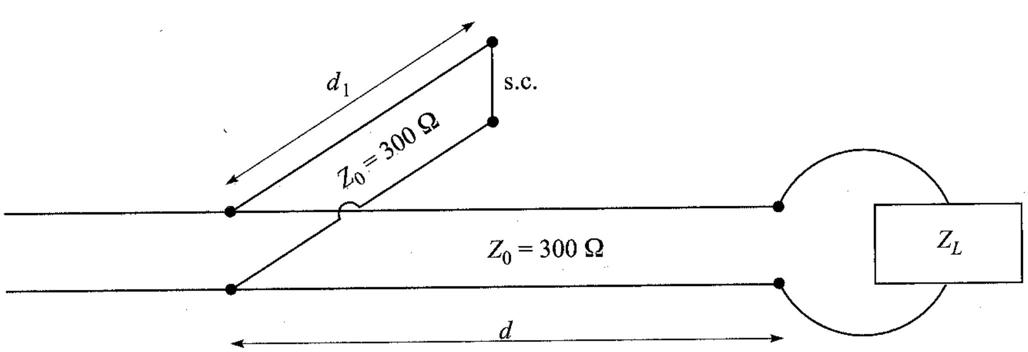


FIGURE 13.32
See Problem 34.

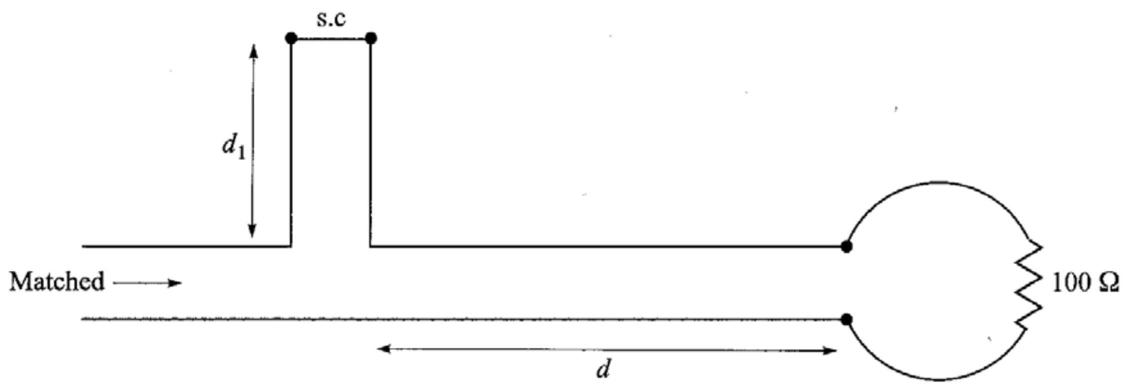


FIGURE 13.33
See Problem 36.

13.37 In the transmission line of Fig. 13.17, $R_L = Z_0 = 50\ \Omega$, and $R_g = 25\ \Omega$. Determine and plot the voltage at the load resistor and the current in the battery as functions of time by constructing appropriate voltage and current reflection diagrams.

13.38 Repeat Problem 37, with $Z_0 = 50\ \Omega$, and $R_L = R_g = 25\ \Omega$. Carry out the analysis for the time period $0 < t < 8l/v$.

13.39 In the transmission line of Fig. 13.17, $Z_0 = 50\ \Omega$, and $R_L = R_g = 25\ \Omega$. The switch is closed at $t = 0$ and *is opened again* at time $t = l/4v$, thus creating a rectangular voltage *pulse* in the line. Construct an appropriate voltage reflection diagram for this case and use it to make a plot of the voltage at the load resistor as a function of time for $0 < t < 8l/v$ (note that the effect of opening the switch is to initiate a second voltage wave, whose value is such that it leaves a net current of zero in its wake).

13.40 In the charged line of Fig. 13.22, the characteristic impedance is $Z_0 = 100\ \Omega$, and $R_g = 300\ \Omega$. The line is charged to initial voltage, $V_0 = 160$ V, and the switch is closed at $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/v$ (four round-trips). This problem accompanies Example 13.6 as the other special case of the basic charged line problem, in which now $R_g > Z_0$.

13.41 In the transmission line of Fig. 13.34, the switch is located *midway* down the line, and is closed at $t = 0$. Construct a voltage reflection diagram for this case, where $R_L = Z_0$. Plot the load resistor voltage as a function of time.

13.42 A simple *frozen wave generator* is shown in Fig. 13.35. Both switches are closed simultaneously at $t = 0$. Construct an appropriate voltage reflection diagram for the case in which $R_L = Z_0$. Determine and plot the load resistor voltage as a function of time.

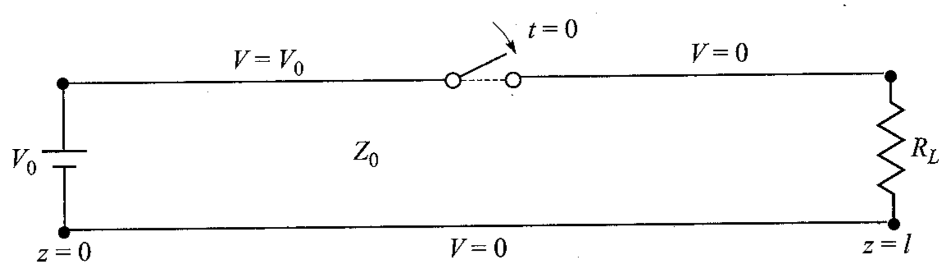


FIGURE 13.34
See Problem 41.

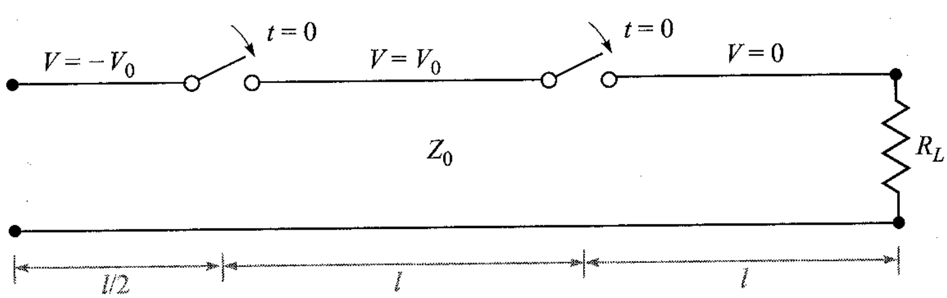


FIGURE 13.35
See Problem 42.